Conditional distribution

$$f\left(\mathbf{y} \mid \mu, \sigma^2\right) = f\left(y_1, y_2 \ldots y_n \mid \mu, \sigma^2\right) = \prod_1^n f\left(y_i \mid \mu, \sigma^2\right) =$$

$$= \prod_1^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(y_i - \mu\right)^2}{2\sigma^2}\right] =$$

$$= \frac{1}{\left(\sqrt{2\pi}\right)^n \left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

$$f\left(\mathbf{y} \mid \mu, \sigma^2\right) = \frac{1}{\left(\sqrt{2\pi}\right)^n \left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

f(σ^2)=constant

$$f\left(\sigma^2 \mid \mathbf{y}, \mu\right) \propto f\left(\mathbf{y} \mid \sigma^2, \mu\right) f\left(\sigma^2\right) \propto \frac{1}{\left(\sqrt{2\pi}\right)^n \left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

NORMAL

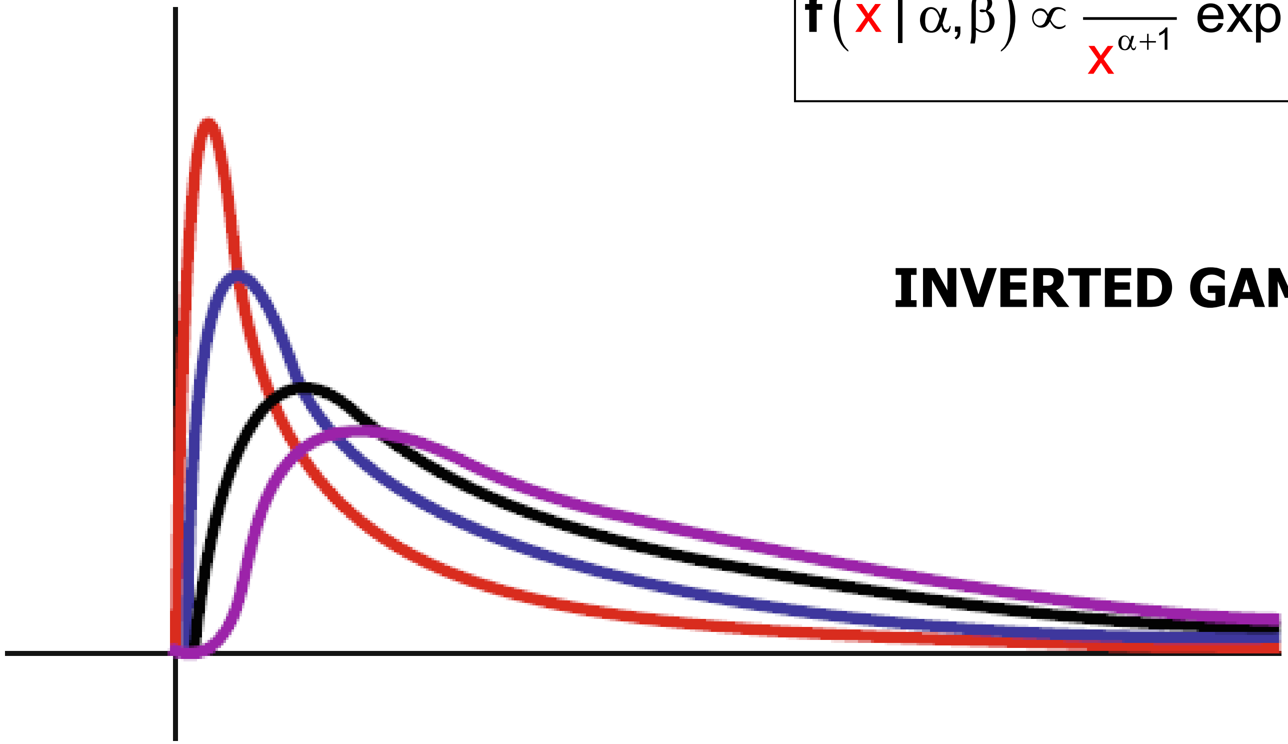
$$f\left(y\right) \propto \frac{1}{\left[\sigma_y^2\right]^{\frac{1}{2}}} \exp\left[-\frac{\left(y - m_y\right)^2}{2\sigma_y^2}\right]$$

NOT NORMAL

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

$$f\left(x \mid \alpha,\beta\right) \propto \frac{1}{x^{\alpha+1}}\exp\left[-\frac{\beta}{x}\right]$$



INVERTED GAMMA

Conditional distribution

$$\mathbf{f}\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

$$\mathbf{f}\left(x \mid \alpha, \beta\right) \propto \frac{1}{x^{\alpha+1}} \exp\left[-\frac{\beta}{x}\right]$$

INVERTED GAMMA

$$\beta = \frac{1}{2}\sum_1^n \left(y_i - \mu\right)^2 \qquad \alpha = \frac{n}{2} - 1$$

Conditional distribution

EXAMPLE

$\mathbf{y}$=[2,6,8]
μ=4

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{3}{2}}} \exp\left[-\frac{(2-4)^2 + (6-4)^2 + (8-4)^2}{2\sigma^2}\right] =$$

$$= \frac{1}{\left(\sigma^2\right)^{\frac{3}{2}}} \exp\left[-\frac{12}{\sigma^2}\right]$$

Conditional distribution

$$f\left(\mu \mid \mathbf{y},\sigma^2\right) \propto f\left(\mathbf{y} \mid \mu,\sigma^2\right) f\left(\mu\right)$$

f(μ)=constant

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i-\mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

NORMAL

$$f\left(y\right) \propto \frac{1}{\left[\sigma_y^2\right]^{\frac{1}{2}}} \exp\left[-\frac{\left(y - m_y\right)^2}{2\sigma_y^2}\right]$$

NOT NORMAL

$$f\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

NORMAL

$$f(\mu) \propto \frac{1}{\left[\operatorname{var}(\mu)\right]^{\frac{1}{2}}} \exp\left[-\frac{\left(\mu - E(\mu)\right)^2}{2\operatorname{var}(\mu)}\right]$$

NOT NORMAL

$$f\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

Conditional distribution

$$f\left(\mu|\sigma^2,\mathbf{y}\right) \propto \exp\left[-\frac{\left(\mu-\bar{y}\right)^2}{2\frac{\sigma^2}{n}}\right] \quad \propto \frac{1}{\sqrt{2\pi}\left(\frac{\sigma^2}{n}\right)^{\frac{1}{2}}}\exp\left[-\frac{\left(\mu-\bar{y}\right)^2}{2\frac{\sigma^2}{n}}\right]$$

$$\boxed{f\left(\mu\,|\,\sigma^2,\mathbf{y}\right) \sim N\left(\bar{y}\,,\frac{\sigma^2}{n}\right)}$$

$$f\left(\mu\right) \propto \frac{1}{\left[\operatorname{var}\left(\mu\right)\right]^{\frac{1}{2}}}\exp\left[-\frac{\left(\mu-E\left(\mu\right)\right)^2}{2\operatorname{var}\left(\mu\right)}\right]$$

Conditional distribution

EXAMPLE

$$f(\mu \mid \sigma^2, \mathbf{y}) \propto \frac{1}{\sqrt{2\pi}\left(\frac{\sigma^2}{n}\right)^{\frac{1}{2}}} \exp\left[-\frac{(\mu-\bar{y})^2}{2\frac{\sigma^2}{n}}\right]$$

$\mathbf{y}$=[2,6,8]

$\sigma^2=2$

$$\mathbf{f}(\mu \mid \sigma^2, \mathbf{y}) \propto \frac{1}{\left(\frac{2}{3}\right)^{\frac{1}{2}}} \exp\left[-\frac{\left(\mu-\frac{2+6+8}{3}\right)^2}{2\cdot\frac{2}{3}}\right]$$

$$= 1.2 \exp\left[-\frac{(\mu-5.3)^2}{1.3}\right]$$

Gibbs sampling

σ^2 arbitrary

$f(\mu|\sigma^2=1,y)$ 4

$f(\sigma^2\,|\mu=4,y)$ 5

$f(\mu|\sigma^2=5,y)$ 2

$f(\sigma^2\,|\mu=2,y)$ 6

σ_0^2 arbitrary

$f(\mu \mid \boldsymbol{\sigma}^2 = \sigma_0^2\ ,\mathbf{y})$

$f(\boldsymbol{\sigma}^2 \mid \mu=\mu_0\ ,\mathbf{y})$

Gibbs sampling

$f(\mu|\sigma^2,\mathbf{y})$: **[**3.1, 3.3, 4.1, 4.8, 4.9, …**]**

$f(\mu|\mathbf{y})$: **[**4.8, 4.9,…**]**

$f(\sigma^2|\mu,\mathbf{y})$: **[**2.4, 2.6, 2.6, 2.6, 2.8,… **]**

$f(\sigma^2|\mathbf{y})$:**[**2.6, 2.8,…**]**

Flat Priors

FIRST: Joint density

$$f\left(\mu,\sigma^2 \mid \mathbf{y}\right) \propto f\left(\mathbf{y} \mid \mu,\sigma^2\right) f\left(\mu,\sigma^2\right) \propto f\left(\mathbf{y} \mid \mu,\sigma^2\right)$$

$f(\mu,\sigma^2) = \text{constant}$

$$f\left(\mu,\sigma^2 \mid \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i-\mu\right)^2}{2\sigma^2}\right]$$

Flat Priors

SECOND: Conditional densities

$$f\left(\mu,\sigma^2 \mid \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i-\mu\right)^2}{2\sigma^2}\right]$$

$$f\left(\mu \mid \sigma^2,\mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i-\mu\right)^2}{2\sigma^2}\right]$$

NORMAL

Algorithms for random sampling

$$f\left(\sigma^2 \mid \mu,\mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i-\mu\right)^2}{2\sigma^2}\right]$$

Inverted gamma

Gibbs sampling

OBJECTIVE: samples of $f(\sigma^2|\mathbf{y})$ and $f(\mu|\mathbf{y})$

σ_0^2 arbitrary

$$f(\mu \mid \boldsymbol{\sigma}^2 = \sigma_0^2\ , \mathbf{y})$$

$$f(\sigma^2 \mid \mu = \mu_0\ , \mathbf{y})$$

Gibbs sampling

$$\mathbf{y}' = [2,4,4,2]$$

$$\bar{y} = 3, \quad n = 4, \quad \sum_{i=1}^{n} y_i = 12, \quad \sum_{1}^{n} y_i^2 = 40$$

$$f\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto N\left(\bar{y}, \frac{\sigma^2}{n}\right) \equiv N\left(3, \frac{\sigma^2}{4}\right)$$

$$\beta = \frac{1}{2}\left[\sum_{1}^{n}\left(y_i - \mu\right)^2\right] = \frac{1}{2}\left[\sum_{1}^{n}\left(y_i^2\right) - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right] = \frac{1}{2}\left(40 - 24\mu + 4\mu^2\right)$$

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \text{Igamma}\begin{cases} \beta = \frac{1}{2}\left[\sum_{1}^{n}\left(y_i - \mu\right)^2\right] \\ \alpha = \frac{n}{2} - 1 \end{cases} \equiv \text{Igamma}\begin{cases} \beta = \frac{1}{2}\left(40 - 24\mu + 4\mu^2\right) \\ \alpha = 1 \end{cases}$$

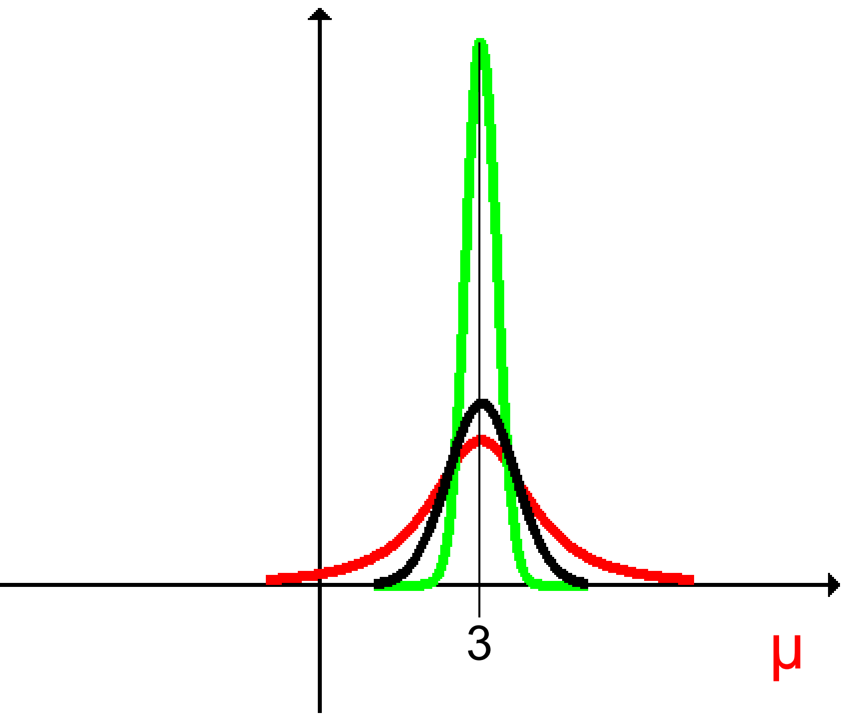
Gibbs sampling

$$f\left(\mu \mid \sigma^2, \mathbf{y}\right) \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right)$$

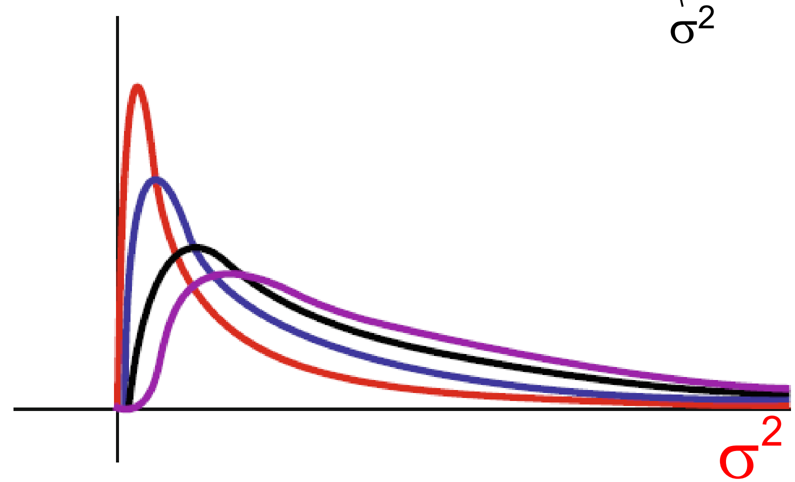
μ

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \sim \text{Igamma}\begin{cases} \beta = \frac{1}{2}\left(40 - 24\mu + 4\mu^2\right) \\ \alpha = 1 \end{cases}$$

σ^2



NORMALS



INVERTED GAMMA

Gibbs sampling

$\mathbf{y}' = [2,4,4,2]$

$\sigma_0^2 = 1$

$\beta = \frac{1}{2}\left(40 - 24\mu + 4\mu^2\right)$

$\mathbf{f}\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto N\left(3, \frac{1}{4}\right)$

$\mu=2$

$\beta = \frac{1}{2}\left(40 - 24 \cdot 2 + 4 \cdot 2^2\right) = 8$

$\mathbf{f}\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \text{Igamma}[8, \ 1]$

$\sigma^2 = 2$

$\beta = \frac{1}{2}\left(40 - 24 \cdot 3 + 4 \cdot 3^2\right) = 4$

$\mathbf{f}\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto N\left(3, \frac{2}{4}\right)$

$\mu=3$

$\mathbf{f}\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto \text{Igamma}[4, \ 1]$

$\sigma^2 = 5$

Gibbs sampling

$N\left(3, \frac{1}{4}\right)$ $N\left(3, \frac{2}{4}\right)$ $N\left(3, \frac{5}{4}\right)$

CONDITIONALS

$f(\mu|\sigma^2,\mathbf{y})$: **[**2, 3, 3.3, 4.1, 4.8, 4.9, …**]**

$f(\mu|\mathbf{y})$: **[**4.8, 4.9,…**]**

MARGINALS

Igamma (8,1) Igamma (4,1) Igamma (7,1)

CONDITIONALS

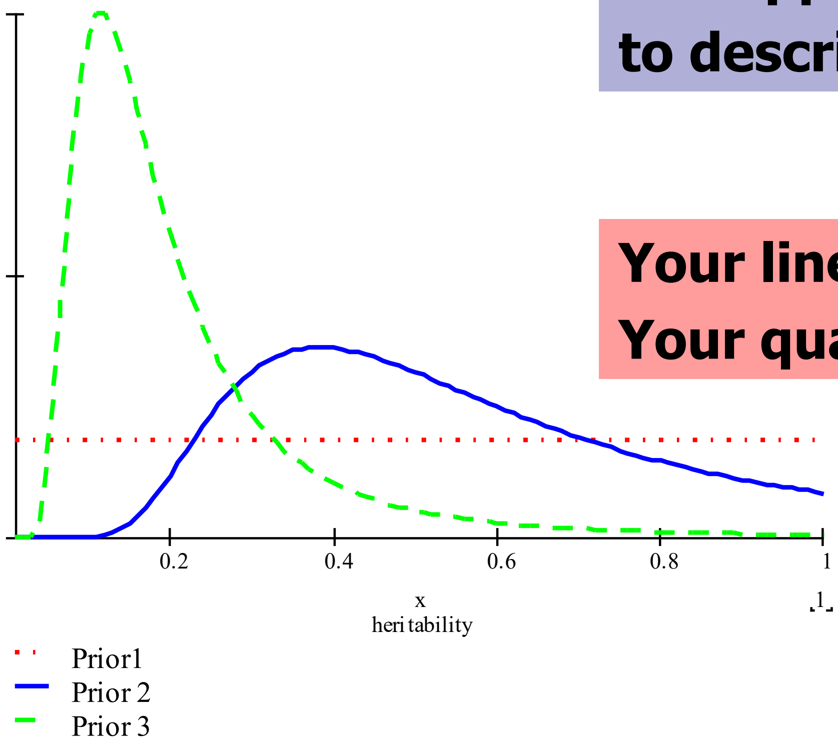
$f(\sigma^2|\mu,\mathbf{y})$: **[**2, 5, 3, 3, 2.6, 2.6, 2.8,... **]**

$f(\sigma^2|\mathbf{y})$: **[**2.6, 2.8,...**]**

MARGINALS

Gibbs sampling
is not a method of estimation
it is a numerical procedure
to obtain
MARGINAL POSTERIOR
DISTRIBUTIONS
used for Bayesian estimation

Vague prior information



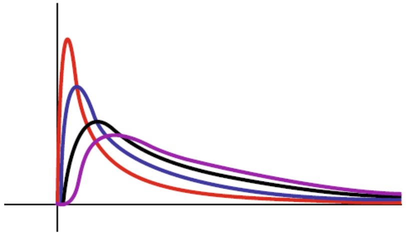
Use appropriate functions to describe vague prior knowledge

Your linear beliefs are symmetrical
Your quadratic beliefs are not

Blasco et al. 1998
Genetics 143: 301-306

Conjugated Priors

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) \propto f\left(\mathbf{y} \mid \sigma^2, \mu\right) f\left(\sigma^2\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n (y_i-\mu)^2}{2\sigma^2}\right] \cdot \frac{1}{\left(\sigma^2\right)^{\alpha+1}} \exp\left[-\frac{\beta}{\sigma^2}\right]$$



$f(\sigma^2)=\mathrm{Igamma}(\beta,\alpha)$

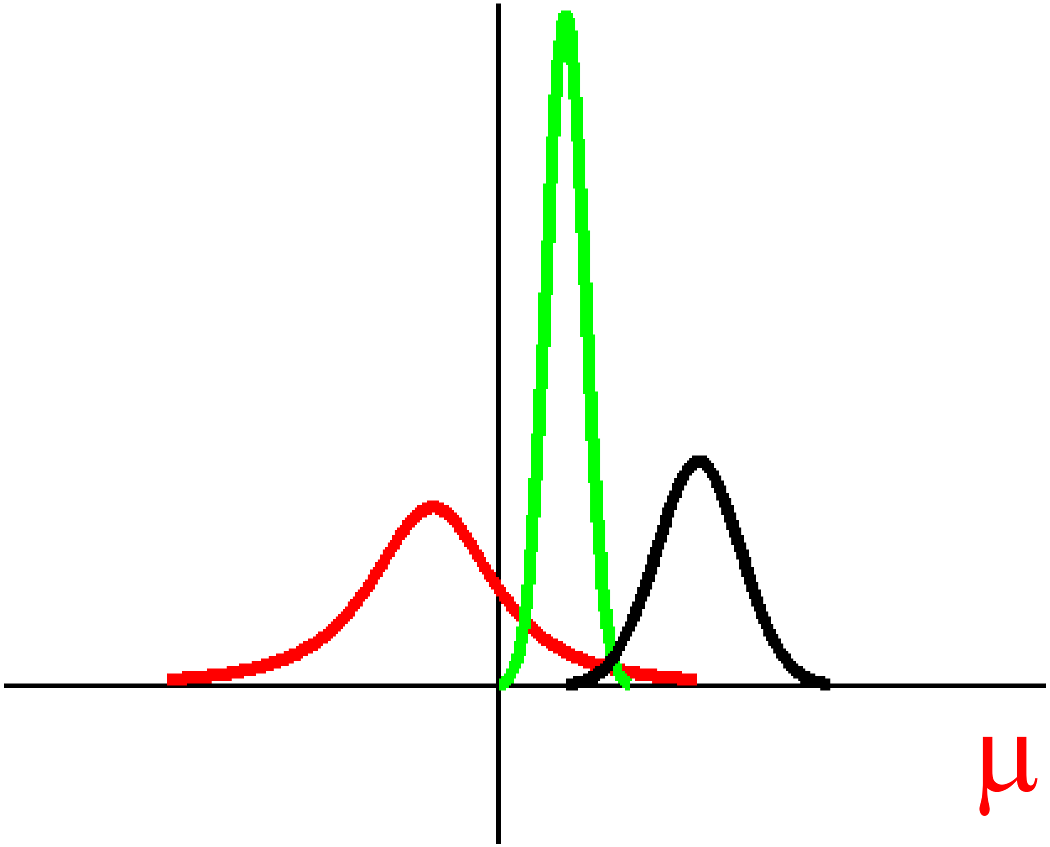
Igamma (b,a)

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}+\alpha+1}} \exp\left[-\frac{\sum_1^n (y_i-\mu)^2 - 2\beta}{2\sigma^2}\right]$$

$$a = \frac{n}{2} + \alpha$$

$$b = \frac{1}{2}\sum_1^n (y_i-\mu)^2 - \beta$$

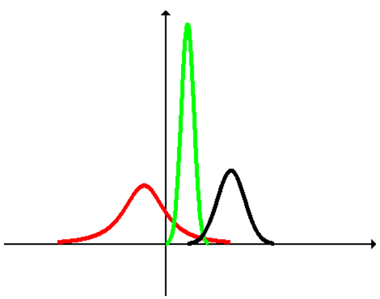
Vague prior information



Conjugated Priors

$$f\left(\mu \mid \sigma^2, \mathbf{y}\right) \propto f\left(\mathbf{y} \mid \mu, \sigma^2\right) f\left(\mu\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_{1}^{n}\left(y_i - \mu\right)^2}{2\sigma^2}\right] \cdot \frac{1}{\left(v^2\right)^{\frac{1}{2}}} \exp\left[-\frac{\left(\mu - m\right)^2}{2v^2}\right]$$

$f(\mu)=N(m,v^2)$



$$\propto \frac{1}{\left(\frac{\sigma^2}{n} \cdot v^2\right)^{\frac{1}{2}}} \exp\left[-\frac{\left(\mu - \bar{y}\right)^2}{2\frac{\sigma^2}{n}} - \frac{\left(\mu - m\right)^2}{2v^2}\right] \propto \exp\left[-\frac{\left(\mu - w\right)^2}{2\frac{\sigma^2}{n} v^2 \frac{1}{v^2 + \frac{\sigma^2}{n}}}\right] \propto N\left(w, d^2\right)$$

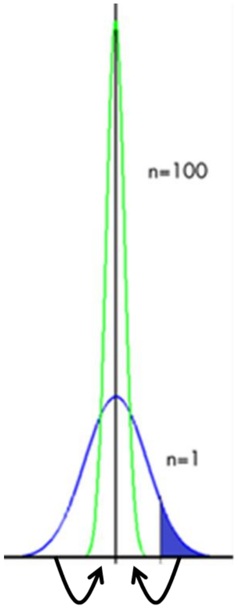
The genetic “mixed” model

THE MODEL

$$y = \mu + S + P + \beta \cdot A + u + p + e$$

Capital: FIXED effects

Small: RANDOM effects



Shrinking

The genetic "mixed" model

THE MODEL

$$y = \mu + S + P + \beta \cdot A + u + p + e$$

cow 1: $7520 = \mu + S_1 + P_1 + \beta \cdot 534 + u_1 + p_1 + e_{11}$
cow 1: $6880 = \mu + S_2 + P_2 + \beta \cdot 534 + u_1 + p_1 + e_{12}$
cow 1: $6920 = \mu + S_3 + P_3 + \beta \cdot 534 + u_1 + p_1 + e_{13}$
cow 2: $9801 = \mu + S_2 + P_1 + \beta \cdot 650 + u_2 + p_2 + e_{21}$
cow 2: $8754 = \mu + S_3 + P_2 + \beta \cdot 650 + u_2 + p_2 + e_{22}$
cow 3: $5950 = \mu + S_1 + P_1 + \beta \cdot 485 + u_2 + p_2 + e_{31}$
... .

The genetic “mixed” model

THE MODEL

$$y = \mu + S + P + \beta \cdot A + u + p + e$$

$$\begin{bmatrix} 7520 \\ 6880 \\ 6920 \\ 9801 \\ 8754 \\ 5950 \\ \cdots \end{bmatrix} = \mathbf{Xb} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \cdots \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \cdots \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{21} \\ e_{22} \\ e_{31} \\ \cdots \end{bmatrix}$$

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Z} \quad \mathbf{u} + \mathbf{W} \quad \mathbf{p} + \mathbf{e}$$

The genetic "mixed" model

THE MODEL

$$y = \mu + S + P + \beta \cdot A + u + p + e$$

$$\begin{bmatrix} 7520 \\ 6880 \\ 6920 \\ 9801 \\ 8754 \\ 5950 \\ \cdots \end{bmatrix} = \mathbf{Xb} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \cdots \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \cdots \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{21} \\ e_{22} \\ e_{31} \\ \cdots \end{bmatrix}$$

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Z} \quad \mathbf{u} + \mathbf{W} \quad \mathbf{p} + \mathbf{e}$$

The genetic “mixed” model

THE MODEL

$$
\mathrm{var}(\mathbf{u}) = \mathrm{var}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \cdots \\ \cdots \end{bmatrix} = \begin{bmatrix}
\sigma_u^2 & \frac{1}{2}\sigma_u^2 & \frac{1}{4}\sigma_u^2 & 0 & 0 & \cdots & \cdots \\
\frac{1}{2}\sigma_u^2 & \sigma_u^2 & \frac{1}{4}\sigma_u^2 & 0 & 0 & \cdots & \cdots \\
\frac{1}{4}\sigma_u^2 & \frac{1}{4}\sigma_u^2 & \sigma_u^2 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \sigma_u^2 & \frac{1}{2}\sigma_u^2 & \cdots & \cdots \\
0 & 0 & 0 & \frac{1}{2}\sigma_u^2 & \sigma_u^2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix} = \mathbf{A}\sigma_u^2
$$

FULL SIBS

HALF SIBS

SIRE-DAUGHTER

The genetic "mixed" model

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Zu} + \mathbf{Wp} + \mathbf{e}$$

OBJECTIVE

$$f(\mathbf{b} \mid \mathbf{y})$$
$$f(\mathbf{u} \mid \mathbf{y})$$
$$f(\mathbf{p} \mid \mathbf{y})$$
$$f(\sigma_u^2 \mid \mathbf{y})$$
$$f(\sigma_p^2 \mid \mathbf{y})$$
$$f(\sigma_e^2 \mid \mathbf{y})$$

First: Joint density

$$f(\mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2 \mid \mathbf{y})$$

Second: Conditionals

Third: Gibbs sampling

$$f(\mathbf{b} \mid \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y})$$
$$f(\mathbf{u} \mid \mathbf{b}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y})$$
$$f(\mathbf{p} \mid \mathbf{b}, \mathbf{u}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y})$$
$$f(\sigma_u^2 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_p^2, \sigma_e^2, \mathbf{y})$$
$$f(\sigma_p^2 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_e^2, \mathbf{y})$$
$$f(\sigma_e^2 \mid \mathbf{u}, \mathbf{c}, \mathbf{b}, \sigma_u^2, \sigma_p^2, \mathbf{y})$$

Gibbs sampling

FIRST: Joint density

$$f\left(\mathbf{b},\mathbf{u},\mathbf{p},\sigma_u^2,\sigma_p^2,\sigma_e^2 \mid \mathbf{y}\right) \propto f\left(\mathbf{y} \mid \mathbf{b},\mathbf{u},\mathbf{p},\sigma_u^2,\sigma_p^2,\sigma_e^2\right)\cdot f\left(\mathbf{b},\mathbf{u},\mathbf{p},\sigma_u^2,\sigma_p^2,\sigma_e^2\right)$$

$$\propto f\left(\mathbf{y} \mid \mathbf{b},\mathbf{u},\mathbf{p},\sigma_u^2,\sigma_p^2,\sigma_e^2\right)\cdot f\left(\mathbf{u} \mid \mathbf{A},\sigma_u^2\right)\cdot f\left(\mathbf{p} \mid \sigma_p^2\right)$$

$$\propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}}\exp\left[-\frac{1}{2\sigma_e^2}\left(\mathbf{y}-\mathbf{X}\mathbf{b}-\mathbf{Z}\mathbf{u}-\mathbf{W}\mathbf{p}\right)'\left(\mathbf{y}-\mathbf{X}\mathbf{b}-\mathbf{Z}\mathbf{u}-\mathbf{W}\mathbf{p}\right)\right]\frac{1}{\left(\sigma_u^2\right)^{\frac{q}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right]\frac{1}{\left(\sigma_p^2\right)^{\frac{m}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]$$

Gibbs sampling

SECOND: Conditionals

$$\propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})\right] \cdot \exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right] \cdot \exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]$$

Gibbs sampling

SECOND: Conditional densities

$$f\left(\sigma_u^2 \mid \mathbf{p},\mathbf{b},\mathbf{u},\sigma_p^2,\sigma_e^2,\mathbf{y}\right) \propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}}\exp\left[-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})\right]\frac{1}{\left(\sigma_u^2\right)^{\frac{q}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right]\frac{1}{\left(\sigma_p^2\right)^{\frac{m}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]\sim IG(\alpha,\beta)$$

$$f\left(\sigma_p^2 \mid \mathbf{p},\mathbf{b},\mathbf{u},\sigma_u^2,\sigma_e^2,\mathbf{y}\right) \propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}}\exp\left[-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})\right]\frac{1}{\left(\sigma_u^2\right)^{\frac{q}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right]\frac{1}{\left(\sigma_p^2\right)^{\frac{m}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]\sim IG(\alpha,\beta)$$

$$f\left(\sigma_e^2 \mid \mathbf{p},\mathbf{b},\mathbf{u},\sigma_u^2,\sigma_p^2,\mathbf{y}\right) \propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}}\exp\left[-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})\right]\frac{1}{\left(\sigma_u^2\right)^{\frac{q}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right]\frac{1}{\left(\sigma_p^2\right)^{\frac{m}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]\sim IG(\alpha,\beta)$$

$$f\left(\mathbf{b},\mathbf{u},\mathbf{p} \mid \sigma_u^2,\sigma_p^2,\sigma_e^2,\mathbf{y}\right) \propto \frac{1}{\left(\sigma_e^2\right)^{\frac{n}{2}}}\exp\left[-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}-\mathbf{Wp})\right]\frac{1}{\left(\sigma_u^2\right)^{\frac{q}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}\right]\frac{1}{\left(\sigma_p^2\right)^{\frac{m}{2}}}\cdot\exp\left[-\frac{1}{2\sigma_p^2}\mathbf{p}'\mathbf{p}\right]\sim N(\mathbf{m},\mathbf{V})$$

A. Blasco. *Bayesian Data Análisis for Animal Scientists*. Appendix 7.1 p. 164

Gibbs sampling

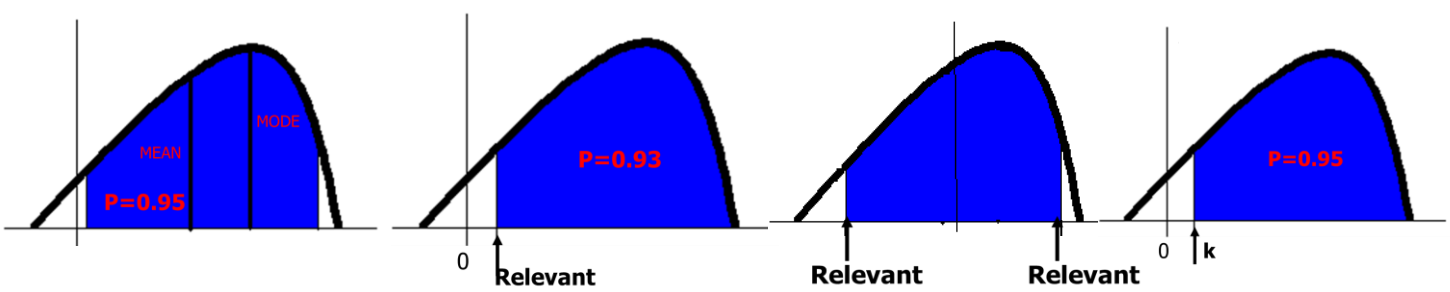
THIRD: Gibbs Sampling

$\mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2$ arbitrary

Conditional	Sampled	Distribution
$f\left(\mathbf{b} \mid \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y}\right)$	$\mathbf{b}$	Normal (multivariate)
$f\left(\mathbf{u} \mid \mathbf{b}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y}\right)$	$\mathbf{u}$	Normal (multivariate)
$f\left(\mathbf{p} \mid \mathbf{b}, \mathbf{u}, \sigma_u^2, \sigma_p^2, \sigma_e^2, \mathbf{y}\right)$	$\mathbf{p}$	Normal (multivariate)
$f\left(\sigma_u^2 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_p^2, \sigma_e^2, \mathbf{y}\right)$	σ_u^2	Igamma
$f\left(\sigma_p^2 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_e^2, \mathbf{y}\right)$	σ_p^2	Igamma
$f\left(\sigma_e^2 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \sigma_u^2, \sigma_p^2, \mathbf{y}\right)$	σ_e^2	Igamma

Gibbs sampling

Muestra	a_1	a_2	…	a_n	σ^2_A	σ^2_e	σ^2_P	h^2	R
1	0.30	-0.05		0.20	0.01	0.05	0.06	0.17	0.41
2	0.05	-0.10		0.22	0.04	0.07	0.11	0.36	0.35
3	0.12	0.12		0.15	0.06	0.12	0.18	0.33	0.29
4	-0.01	-0.35		0.35	0.06	0.09	0.12	0.50	0.31
…	…	…		…	…	…	…	…	…



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BLUP as a Bayesian estimator

HOW HENDERSON DERIVED BLUP (Henderson, 1949)

https://acteon.webs.upv.es/ARTICULOS/HENDERSON%201949.pdf

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Zu} + \mathbf{e}$$

$$\boxed{E(\mathbf{y} \mid \mathbf{b},\mathbf{u}) = \mathbf{Xb} + \mathbf{Zu}}$$

Henderson says he maximizes

$$\varphi = f(\mathbf{y} \mid \mathbf{u})\, f(\mathbf{u})$$

$$\propto \exp\left[-\frac{1}{2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu})'\left(\mathbf{I}\sigma_e^2\right)^{-1}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu})\right]\exp\left[-\frac{1}{2}\mathbf{u}'\mathbf{G}^{-1}\mathbf{u}\right]$$

This is really $f(\mathbf{y} \mid \mathbf{u},\mathbf{b},\sigma_e^2)$

where $\mathbf{G} = \mathbf{A}\sigma_u^2$

BLUP as a Bayesian estimator

WHAT HENDERSON REALLY DID

Henderson maximized

If **u** and **b** are independent

$$\varphi = f\left(\mathbf{u},\mathbf{b} \mid \mathbf{y},\mathbf{G},\sigma_e^2\right) \propto f\left(\mathbf{y} \mid \mathbf{u},\mathbf{b},\mathbf{G},\sigma_e^2\right) f\left(\mathbf{u},\mathbf{b} \mid \mathbf{G},\sigma_e^2\right) \propto$$

$\mathbf{G} = \mathbf{A}\sigma_u^2$

If f(**b**)=constant

$$\propto f\left(\mathbf{y} \mid \mathbf{u},\mathbf{b},\mathbf{G},\sigma_e^2\right) f\left(\mathbf{u} \mid \mathbf{G},\sigma_e^2\right) f\left(\mathbf{b}\right) \propto$$

$$\propto f\left(\mathbf{y} \mid \mathbf{u},\mathbf{b},\mathbf{G},\sigma_e^2\right) f\left(\mathbf{u} \mid \mathbf{G},\sigma_e^2\right) \propto$$

$$\propto \exp\left[-\frac{1}{2}(\mathbf{y}-\mathbf{X}\mathbf{b}-\mathbf{Z}\mathbf{u})'\left(\mathbf{I}\sigma_e^2\right)^{-1}(\mathbf{y}-\mathbf{X}\mathbf{b}-\mathbf{Z}\mathbf{u})\right]\exp\left[-\frac{1}{2}\mathbf{u}'\mathbf{G}^{-1}\mathbf{u}\right]$$

$$\mathbf{G} = \mathbf{A}\sigma_u^2\ ; \quad \mathbf{G}^{-1} = \frac{1}{\sigma_u^2}\mathbf{A}^{-1}$$

BLUP as a Bayesian estimator

$$\log \varphi \propto \frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu})'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu})+\frac{1}{2\sigma_u^2}\mathbf{u}'\mathbf{A}^{-1}\mathbf{u}$$

$$\frac{\partial}{\partial \mathbf{b}}\ln\varphi \propto -2\frac{1}{2\sigma_e^2}\mathbf{X}'(\mathbf{y}-\mathbf{Xb})=0$$

$$\frac{\partial}{\partial \mathbf{u}}\ln\varphi \propto -2\frac{1}{2\sigma_e^2}\mathbf{Z}'(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu})+2\frac{1}{2\sigma_u^2}\mathbf{A}^{-1}\mathbf{u}=0$$

Equating to zero to find the maxima, we obtain

$$\begin{aligned}\mathbf{X}'\mathbf{X}\hat{\mathbf{b}}+\mathbf{X}'\mathbf{Z}\hat{\mathbf{u}}&=\mathbf{X}'\mathbf{y}\\ \mathbf{Z}'\mathbf{X}\hat{\mathbf{b}}+\mathbf{Z}'\mathbf{Z}\hat{\mathbf{u}}+\alpha\mathbf{A}^{-1}\hat{\mathbf{u}}&=\mathbf{Z}'\mathbf{y}\end{aligned} \quad \rightarrow \quad \begin{bmatrix}\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z}\\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z}+\alpha\mathbf{A}^{-1}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{b}}\\ \hat{\mathbf{u}}\end{bmatrix}=\begin{bmatrix}\mathbf{X}'\mathbf{y}\\ \mathbf{Z}'\mathbf{y}\end{bmatrix}$$

$$\boxed{\text{where } \alpha=\frac{\sigma_e^2}{\sigma_u^2}}$$

BLUP as a Bayesian estimator

$$\begin{bmatrix} \mathbf{X'X} & \mathbf{X'Z} \\ \mathbf{Z'X} & \mathbf{Z'Z} + \alpha\mathbf{A}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{X'y} \\ \mathbf{Z'y} \end{bmatrix}$$

$$\text{if } \alpha = \frac{\sigma_e^2}{\sigma_u^2} \longrightarrow 0 \quad \text{then } \mathbf{u} \text{ is "fixed"}$$

$$\text{if } \sigma_u^2 \longrightarrow \infty \quad \text{then } \mathbf{u} \text{ is "fixed"}$$

$$\text{if } \sigma_u^2 \longrightarrow \infty \quad \text{then } f(\mathbf{u}) = \text{constant}$$

Fixed effects are the same as random effects, but with different priors

BLUP as a Bayesian estimator

$$\boxed{\begin{array}{l}\mathrm{BLUP} = \text{mode } f\left(\mathbf{u},\mathbf{b} \mid \mathbf{A},\sigma_u^2,\sigma_e^2,\mathbf{y}\right) \\ \text{for independent } \mathbf{u} \text{ and } \mathbf{b} \text{ and } f\left(\mathbf{b}\right) = \text{constant}\end{array}}$$

Notice that here BLUP is not unbiased
(there is no bias in a Bayesian context)

Notice that here σ_u^2, σ_e^2 "true values" are not required

Notice that here b is not "fixed", it only has a different prior
(there are not fixed effects in a Bayesian context)

Notice that the usual Bayesian solution is better than BLUP
(posterior densities of all unknowns, including variances)

Interlude

Inference on breeding values and genetic parameters under selection

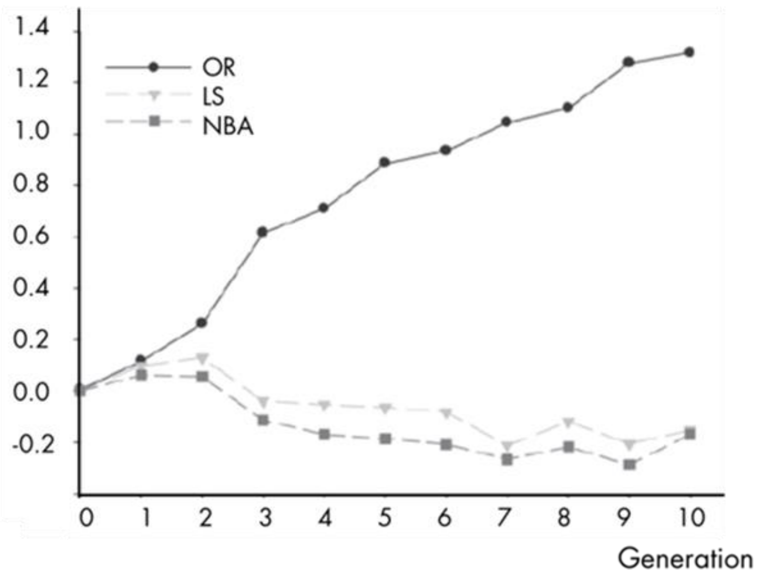
Selected data

$$\mathbf{y} = \mathbf{Xb} + \mathbf{Zu} + \mathbf{e}$$

$$f\left(\mathbf{u},\mathbf{b} \mid \mathbf{y},\mathbf{G},\sigma_e^2\right) \propto \exp\left[-\frac{1}{2}\left(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}\right)'\left(\mathbf{I}\sigma_e^2\right)^{-1}\left(\mathbf{y}-\mathbf{Xb}-\mathbf{Zu}\right)\right]\exp\left[-\frac{1}{2}\mathbf{u}'\mathbf{G}^{-1}\mathbf{u}\right]$$

$\boxed{E(\mathbf{u}) = \mathbf{0}}$

$E(\mathbf{u}) \neq \mathbf{0}$ when data are selected



BLUP: ignore selection if a mysterious **L** gives **L'X=0**

REML: ignore selection if all data used in selection and full relationship matrix are included

	British (B)	Spanish (S)
Men (M)	40%	30%
Women (W)	20%	10%

$$f(BM,\ BW,\ SM,\ SW) = (0.4,\ 0.2,\ 0.3,\ 0.1)$$

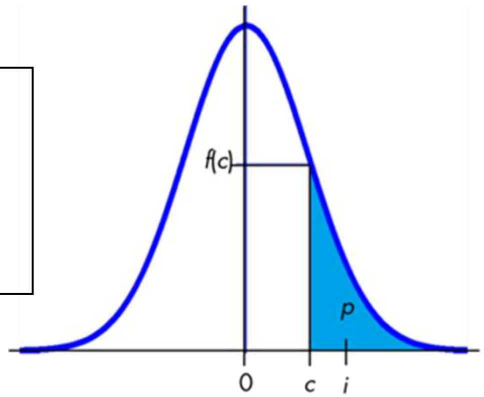
$$f(M,B) = f(M \mid B) \cdot f(B) \quad \rightarrow \quad f(M \mid B) = \frac{f(M,B)}{f(B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$f(W,B) = f(W \mid B) \cdot f(B) \quad \rightarrow \quad f(W \mid B) = \frac{f(W,B)}{f(B)} = \frac{0.2}{0.6} = \frac{1}{3}$$

$$f_S(BM,\ BW) = f(M \mid B\ ,\ W \mid B) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Selected data

$\mathbf{y}_0$: Data from generation 0, not selected
$\mathbf{y}_1$: Data from generation 1 from selected parents of generation 0
c : Selection threshold
$\boldsymbol{\theta}$: Parameters and effects that we have to estimate



$$f_S(\mathbf{y}_1,\mathbf{y}_0) = f(\mathbf{y}_1,\mathbf{y}_0 \mid \mathbf{y}_0 > c) = \frac{f(\mathbf{y}_1,\mathbf{y}_0)}{f(\mathbf{y}_0 > c)}$$

$$\boxed{f(x \mid y) = \frac{f(x,y)}{f(y)}}$$

$$f_S(\mathbf{y}_1,\mathbf{y}_0,\theta) = f(\mathbf{y}_1,\mathbf{y}_0,\theta \mid \mathbf{y}_0 > c) = \frac{f(\mathbf{y}_1,\mathbf{y}_0,\theta)}{f(\mathbf{y}_0 > c)}$$

$$f_S(\mathbf{y}_1,\mathbf{y}_0,\theta) = f_S(\theta \mid \mathbf{y}_1,\mathbf{y}_0) f_S(\mathbf{y}_1,\mathbf{y}_0)$$

$$\boxed{f(x,y) = f(x \mid y) f(y)}$$

$$\boxed{f_S(\theta \mid \mathbf{y}_1,\mathbf{y}_0)} = \frac{f_S(\mathbf{y}_1,\mathbf{y}_0,\theta)}{f_S(\mathbf{y}_1,\mathbf{y}_0)} = \frac{\dfrac{f(\mathbf{y}_1,\mathbf{y}_0,\theta)}{f(\mathbf{y}_0 > c)}}{\dfrac{f(\mathbf{y}_1,\mathbf{y}_0)}{f(\mathbf{y}_0 > c)}} = \frac{f(\mathbf{y}_1,\mathbf{y}_0,\theta)}{f(\mathbf{y}_1,\mathbf{y}_0)} = \boxed{f(\theta \mid \mathbf{y}_1,\mathbf{y}_0)}$$

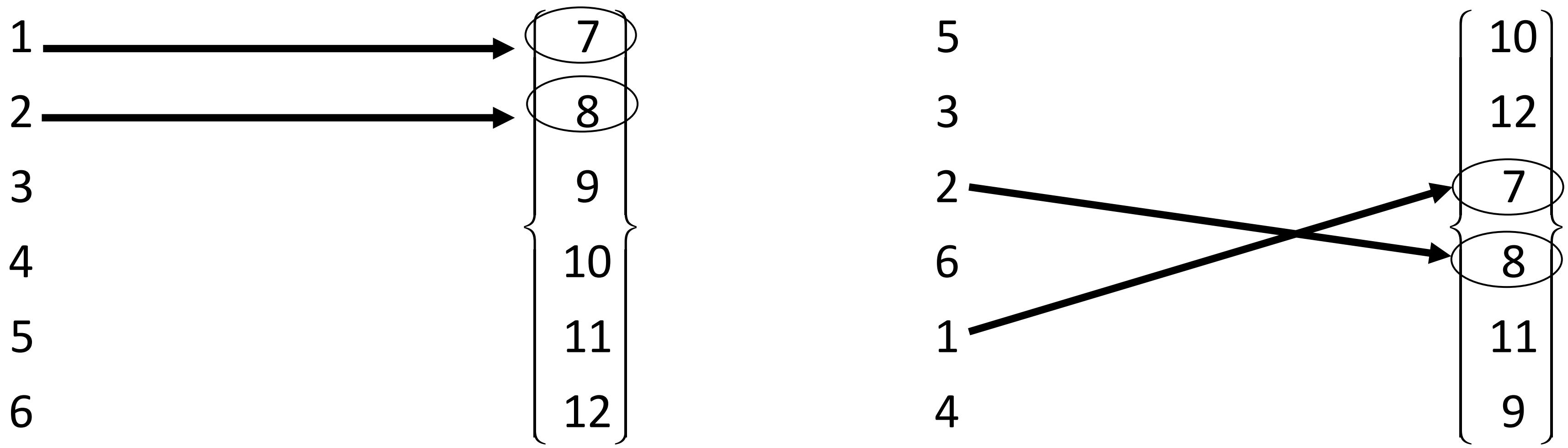
Selected data

mutatis mutandi

$$f_S(\boldsymbol{\theta} \mid \boldsymbol{y}^D, \boldsymbol{y}^C) = f(\boldsymbol{\theta} \mid \boldsymbol{y}^D, \boldsymbol{y}^C)$$

$\boldsymbol{y}^D$ = data directly selected
$\boldsymbol{y}^C$ = correlated data, indirectly selected

Selected data



Repetition 1

Repetition 2

Selected data

- **10.151** rabbits of 11 generations of selection, with single data of growth rate $\mathbf{y}^D$

- **137** rabbits of generation 11, weekly weighted from weeks 1 to 40 (not all of them arrived to the 40th week of age)

but we estimate the unknowns of all the rabbits !

(Data augmentation)

End of the Interlude

The multivariate model

The multivariate model

$$\mathbf{y}_1 = \mathbf{X}_1\mathbf{b}_1 + \mathbf{e}_1$$

$$\mathbf{y}_2 = \mathbf{X}_2\mathbf{b}_2 + \mathbf{W}\mathbf{c} + \mathbf{e}_2$$

$$\mathbf{b}_1 \sim \text{constant}$$

$$\mathbf{b}_2 \sim \text{constant}$$

$$\mathbf{c} \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{c}\right)$$

$$\mathbf{e}_1 \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{e1}\right)$$

$$\mathbf{e}_2 \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{e2}\right)$$

$$\mathbf{R}_{\substack{\text{SORTED BY}\\ \text{INDIVIDUAL}}} = \mathbf{I} \otimes \mathbf{R}_0$$

$$\mathbf{C}_0 = \begin{bmatrix} \sigma^2_{c_1} & 0 \\ 0 & \sigma^2_{e_2} \end{bmatrix}$$

$$\mathbf{R}_0 = \begin{bmatrix} \sigma^2_{e_1} & \sigma_{e_1e_2} \\ \sigma_{e_1e_2} & \sigma^2_{e_2} \end{bmatrix}$$

The multivariate model

DATA AUGMENTATION

$$\mathbf{y}_1^* = \begin{bmatrix} \mathbf{y_1} \\ \mathbf{z_1} \end{bmatrix} \qquad \mathbf{y}_2^* = \begin{bmatrix} \mathbf{y}_2 \\ \mathbf{z}_2 \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{X}_1\mathbf{b}_1 + \mathbf{e}_1$$
$$\mathbf{y}_2 = \mathbf{X}_2\mathbf{b}_2 + \mathbf{Wc} + \mathbf{e}_2$$

$$\mathbf{y}_1^* = \mathbf{Xb}_1 + \mathbf{Wc}_1 + \mathbf{e}_1$$
$$\mathbf{y}_2^* = \mathbf{Xb}_2 + \mathbf{Wc}_2 + \mathbf{e}_2$$

$$\mathbf{C}_0 = \begin{bmatrix} \sigma_{c_1}^2 & 0 \\ 0 & \sigma_{c_2}^2 \end{bmatrix}$$

$$\mathbf{R}_0 = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1e_2} \\ \sigma_{e_1e_2} & \sigma_{e_2}^2 \end{bmatrix}$$

$$\mathbf{y^*} = \mathbf{Xb} + \mathbf{Wc} + \mathbf{e}$$

$\mathbf{b} \sim$ constant

$\mathbf{c} \sim N(\mathbf{0}, \mathbf{I} \otimes \mathbf{C}_0)$

$\mathbf{e} \sim N(\mathbf{0}, \mathbf{I} \otimes \mathbf{R}_0)$

$$\boldsymbol{\theta} = \mathbf{b}, \mathbf{c}, \mathbf{C}_0, \mathbf{R}_0$$

The multivariate model

DATA AUGMENTATION

$$\mathbf{y}^* = \mathbf{Xb} + \mathbf{Wc} + \mathbf{e}$$

$$\boxed{\boldsymbol{\theta} = \mathbf{b}, \mathbf{c}, \mathbf{C}_0, \mathbf{R}_0}$$

OUR OBJECTIVE

$$\boxed{f(\mathbf{b} \mid \mathbf{y}^*), f(\mathbf{c} \mid \mathbf{y}^*), f(\sigma^2_{c1} \mid \mathbf{y}^*), f(\sigma^2_{c2} \mid \mathbf{y}^*), f(\sigma^2_{e1} \mid \mathbf{y}^*), f(\sigma^2_{e2} \mid \mathbf{y}^*), f(\sigma^2_{e1e2} \mid \mathbf{y}^*)}$$

OUR UNKNOWNS

$$\boxed{\mathbf{b}, \mathbf{c}, \mathbf{C}_0, \mathbf{R}_0, \mathbf{z} = \boldsymbol{\theta}, \mathbf{z}}$$

The multivariate model

First: Joint distribution

$$\boxed{f\left(\mathbf{b},\mathbf{c},\mathbf{C}_0,\mathbf{R}_0,\mathbf{z}\mid\mathbf{y}\right)=f\left(\boldsymbol{\theta},\mathbf{z}\mid\mathbf{y}\right)}$$

Second: Conditionals

$$f\left(\mathbf{z}\mid\mathbf{c},\mathbf{b},\mathbf{C}_0,\mathbf{R}_0,\mathbf{y}\right)$$

$$f\left(\mathbf{b}\mid\mathbf{c},\mathbf{C}_0,\mathbf{R}_0,\mathbf{y},\mathbf{z}\right)$$

$$f\left(\mathbf{c}\mid\mathbf{b},\mathbf{C}_0,\mathbf{R}_0,\mathbf{y},\mathbf{z}\right)$$

$$f\left(\mathbf{C}_0\mid\mathbf{c},\mathbf{b},\mathbf{R}_0,\mathbf{y},\mathbf{z}\right)$$

$$f\left(\mathbf{R}_0\mid\mathbf{c},\mathbf{b},\mathbf{C}_0,\mathbf{y},\mathbf{z}\right)$$

The multivariate model

$$f\left(\boldsymbol{\theta},\mathbf{z} \mid \mathbf{y}\right) \propto f\left(\mathbf{y} \mid \boldsymbol{\theta},\mathbf{z}\right) \cdot f\left(\boldsymbol{\theta},\mathbf{z}\right) =$$

$$= f\left(\mathbf{y} \mid \boldsymbol{\theta},\mathbf{z}\right) \cdot f\left(\mathbf{z} \mid \boldsymbol{\theta}\right) \cdot f\left(\boldsymbol{\theta}\right)$$

$$\boxed{f\left(\mathbf{y},\mathbf{z} \mid \boldsymbol{\theta}\right) = f\left(\mathbf{y} \mid \mathbf{z},\boldsymbol{\theta}\right) \cdot f\left(\mathbf{z} \mid \boldsymbol{\theta}\right)}$$

$$= f\left(\mathbf{y},\mathbf{z} \mid \boldsymbol{\theta}\right) \cdot f\left(\boldsymbol{\theta}\right) = f\left(\mathbf{y}^{*} \mid \boldsymbol{\theta}\right) \cdot f\left(\boldsymbol{\theta}\right)$$

then by Gibbs sampling we

obtain $\boldsymbol{\theta}$ and $\mathbf{z}$, and we forget $\mathbf{z}$

… all inferences are conditioned ONLY on $\mathbf{y}$

The multivariate model

Conditionals:

$f\left(\mathbf{z} \mid \mathbf{c}, \mathbf{b}, \mathbf{C}_0, \mathbf{R}_0, \mathbf{y}\right)$	Normal (multivariate)
$f\left(\mathbf{b} \mid \mathbf{c}, \mathbf{C}_0, \mathbf{R}_0, \mathbf{y}, \mathbf{z}\right)$	Normal (multivariate)
$f\left(\mathbf{c} \mid \mathbf{b}, \mathbf{C}_0, \mathbf{R}_0, \mathbf{y}, \mathbf{z}\right)$	Normal (multivariate)
$f\left(\mathbf{C}_0 \mid \mathbf{c}, \mathbf{b}, \mathbf{R}_0, \mathbf{y}, \mathbf{z}\right)$	Inverted Wishart
$f\left(\mathbf{R}_0 \mid \mathbf{c}, \mathbf{b}, \mathbf{C}_0, \mathbf{y}, \mathbf{z}\right)$	Inverted Wishart

Inverted Wishart = multivariate Inverted Gamma

The multivariate model

EXAMPLE

G: Growth
F: Food intake

SET1: Growth & Food intake recorded
SET2: Only Growth recorded

$$\mathbf{y}_{G1} = \mathbf{X}_1\mathbf{b}_G + \mathbf{e}_{G1} \qquad \mathbf{y}_{F1} = \mathbf{X}_1\mathbf{b}_F + \mathbf{e}_{F1}$$

$$\mathbf{y}_{G2} = \mathbf{X}_2\mathbf{b}_G + \mathbf{e}_{G2} \qquad \mathbf{z} = \mathbf{X}_2\mathbf{b}_F + \mathbf{e}_{F2}$$

$$\mathbf{y}_G^* = \begin{bmatrix} \mathbf{y}_{G1} \\ \mathbf{y}_{G2} \end{bmatrix} \qquad \mathbf{y}_F^* = \begin{bmatrix} \mathbf{y}_{F1} \\ \mathbf{z} \end{bmatrix}$$

$\mathbf{X}$ is the same for both traits

$$\mathbf{y}^* = \mathbf{Xb} + \mathbf{e}$$

The multivariate model

$$y - m_y = \beta\left(x - m_x\right) + e$$

$$E\left(y \mid x\right) = m_y + \beta\left(x - m_x\right) = m_y + \frac{\operatorname{cov}\left(x, y\right)}{\sigma_X^2}\left(x - m_x\right)$$

$$E\left(z \mid y\right) = m_z + \frac{\operatorname{cov}\left(z, y\right)}{\sigma_y^2}\left(y - m_y\right)$$

$$\begin{aligned} \mathbf{y}_{G2} &= \mathbf{X}_2\mathbf{b}_G + \mathbf{e}_{G2} \\ \mathbf{z} &= \mathbf{X}_2\mathbf{b}_F + \mathbf{e}_{F2} \end{aligned} \qquad \begin{aligned} \mathbf{m}_y &= E\left(\mathbf{y}_{G2}\right) = \mathbf{X}_2\mathbf{b}_G \\ \mathbf{m}_z &= E\left(\mathbf{z}\right) = \mathbf{X}_2\mathbf{b}_F \end{aligned} \qquad \frac{\operatorname{cov}\left(z, y\right)}{\sigma_y^2} = \frac{\operatorname{cov}\left(e_G, e_F\right)}{\sigma_{e_G}^2}$$

Gibbs sampling

z and **y** are jointly Normally distributed

$$E(z \mid y) = m_z + \frac{\mathrm{cov}(z,y)}{\sigma_y^2}(y - m_y)$$

$$\mathbf{z} \mid \mathbf{b}_G, \mathbf{b}_F, \mathbf{R}, \mathbf{y}_G, \mathbf{y}_F \sim N\left[E(\mathbf{z} \mid \mathbf{y}), \mathrm{Var}(\mathbf{z} \mid \mathbf{y})\right]$$

$$\mathbf{z} \mid \mathbf{b}_G, \mathbf{b}_F, \mathbf{R}, \mathbf{y}_G, \mathbf{y}_F \sim N\left[\mathbf{X}_2\mathbf{b}_F + \frac{\sigma_{e_G e_F}}{\sigma_{e_G}^2}(\mathbf{y}_{G2} - \mathbf{X}_2\mathbf{b}_G),\ \mathbf{I}\left(\sigma_{e_F}^2 - \frac{\sigma_{e_G e_F}^2}{\sigma_{e_G}^2}\right)\right]$$

Gibbs sampling

$$f\left(\mathbf{z} \mid \mathbf{c}, \mathbf{b}, \sigma_c^2, \sigma_e^2, \mathbf{y}\right) \qquad \boxed{\mathbf{z}}$$

$$f\left(\mathbf{b} \mid \mathbf{c}, \sigma_c^2, \sigma_e^2, \mathbf{y}, \mathbf{z}\right) \qquad \boxed{\mathbf{b}}$$

$$f\left(\mathbf{c} \mid \mathbf{b}, \sigma_c^2, \sigma_e^2, \mathbf{y}, \mathbf{z}\right) \qquad \boxed{\mathbf{c}}$$

$$f\left(\sigma_c^2 \mid \mathbf{c}, \mathbf{b}, \sigma_e^2, \mathbf{y}, \mathbf{z}\right) \qquad \boxed{\sigma_c^2}$$

$$f\left(\sigma_e^2 \mid \mathbf{c}, \mathbf{b}, \sigma_c^2, \mathbf{y}, \mathbf{z}\right) \qquad \boxed{\sigma_e^2}$$

The multivariate model

Magic !!

The genetic multivariate model

$$\mathbf{y}_1 = \mathbf{X}_1\mathbf{b}_1 + \mathbf{Z}_1\mathbf{u}_1 + \mathbf{e}_1$$

$$\mathbf{y}_2 = \mathbf{X}_2\mathbf{b}_2 + \mathbf{Z}_2\mathbf{u}_2 + \mathbf{W}\mathbf{p} + \mathbf{e}_2$$

$$\mathbf{b}_1 \sim \text{constant}$$

$$\mathbf{b}_2 \sim \text{constant}$$

$$\mathbf{u}_1 \mid \mathbf{A} \sim N\left(\mathbf{0}, \mathbf{A}\sigma^2_{u1}\right)$$

$$\mathbf{u}_2 \mid \mathbf{A} \sim N\left(\mathbf{0}, \mathbf{A}\sigma^2_{u2}\right)$$

$$\mathbf{p} \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{p}\right)$$

$$\mathbf{e}_1 \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{e1}\right)$$

$$\mathbf{e}_2 \sim N\left(\mathbf{0}, \mathbf{I}\sigma^2_{e2}\right)$$

$$\text{cov}\left(\mathbf{e}_1, \mathbf{e}_2\right)_{\substack{\text{SORTED BY}\\ \text{INDIVIDUAL}}} = \mathbf{I} \otimes \mathbf{R}_0$$

$$\text{cov}\left(\mathbf{u}_1, \mathbf{u}_2\right)_{\substack{\text{SORTED BY}\\ \text{INDIVIDUAL}}} = \mathbf{A} \otimes \mathbf{G}_0$$

$$\mathbf{R}_0 = \begin{bmatrix} \sigma^2_{e_1} & \sigma_{e_1e_2} \\ \sigma_{e_1e_2} & \sigma^2_{e_2} \end{bmatrix} \qquad \mathbf{P}_0 = \begin{bmatrix} \sigma^2_{p_1} & 0 \\ 0 & \sigma^2_{p_2} \end{bmatrix}$$

$$\mathbf{G}_0 = \begin{bmatrix} \sigma^2_{u_1} & \sigma_{u_1u_2} \\ \sigma_{u_1u_2} & \sigma^2_{u_2} \end{bmatrix}$$

The genetic multivariate model

DATA AUGMENTATION

$$\mathbf{y}_1^* = \mathbf{X}\mathbf{b}_1 + \mathbf{Z}\mathbf{u}_1 + \mathbf{W}\mathbf{p}_1 + \mathbf{e}_1$$

$$\mathbf{y}_2^* = \mathbf{X}\mathbf{b}_2 + \mathbf{Z}\mathbf{u}_2 + \mathbf{W}\mathbf{p}_2 + \mathbf{e}_2$$

$$\mathbf{y}_1^* = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{z}_1 \end{bmatrix} \qquad \mathbf{y}_2^* = \begin{bmatrix} \mathbf{y}_2 \\ \mathbf{z}_2 \end{bmatrix}$$

$$\mathbf{y}^* = \mathbf{X}\mathbf{b} + \mathbf{Z}\mathbf{u} + \mathbf{W}\mathbf{p} + \mathbf{e}$$

$$\boxed{\boldsymbol{\theta} = \mathbf{b}, \mathbf{u}, \mathbf{p}, \mathbf{G}_0, \mathbf{P}_0, \mathbf{R}_0}$$

Gibbs sampling

$$f\left(\mathbf{z} \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \mathbf{G}_0, \mathbf{R}_0, \mathbf{P}_0, \mathbf{y}\right) \quad \boxed{\mathbf{z}}$$

$$f\left(\mathbf{b} \mid \mathbf{u}, \mathbf{p}, \mathbf{G}_0, \mathbf{R}_0, \mathbf{P}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{b}}$$

$$f\left(\mathbf{u} \mid \mathbf{b}, \mathbf{p}, \mathbf{G}_0, \mathbf{R}_0, \mathbf{P}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{u}}$$

$$f\left(\mathbf{p} \mid \mathbf{b}, \mathbf{u}, \mathbf{G}_0, \mathbf{R}_0, \mathbf{P}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{p}}$$

$$f\left(\mathbf{G}_0 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \mathbf{R}_0, \mathbf{P}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{G}_0}$$

$$f\left(\mathbf{R}_0 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \mathbf{G}_0, \mathbf{P}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{R}_0}$$

$$f\left(\mathbf{P}_0 \mid \mathbf{b}, \mathbf{u}, \mathbf{p}, \mathbf{G}_0, \mathbf{R}_0, \mathbf{y}, \mathbf{z}\right) \quad \boxed{\mathbf{P}_0}$$

Gibbs sampling

Muestra	u_1	u_2	…	u_n	σ_u^2	σ_e^2	σ_P^2	h^2	R
1	0.30	-0.05		0.20	0.01	0.05	0.06	0.17	0.41
2	0.05	-0.10		0.22	0.04	0.07	0.11	0.36	0.35
3	0.12	0.12		0.15	0.06	0.12	0.18	0.33	0.29
4	-0.01	-0.35		0.35	0.06	0.09	0.12	0.50	0.31
…	…	…		…	…	…	…	…	…