Modelling Epidemics

Lecture 5: Deterministic compartmental epidemiological models in homogeneous populations

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Overview

• Key characteristics of epidemics
  • What is an epidemic and what do we need to know about them?
  • The basic reproductive number $R_0$

• Modelling epidemics: basic compartmental models
  • Deterministic model formulation
  • SIR model without demography
  • SIR model with demography

• Adding complexity
  • Loss of immunity
  • Inclusion of chronic carriers
What is an epidemic?

Definition: Epidemic

A widespread occurrence of an infectious disease in a community at a particular time.
Questions for modelling epidemics

• What is the risk of an epidemic to occur?
• How severe is the epidemic?
  • What proportion of the population will become infected?
  • What proportion will die?
• How long will it last?
• Are all individuals at risk of becoming infected?
• How far will it spread?
• What impact does a particular intervention have on the risk, severity and duration of the epidemic?
The basic reproductive ratio $R_0$

- $R_0$ is a key epidemiological measure for how “infectious” a disease is.

**Definition:** Basic reproductive ratio $R_0$

The average number of people an infectious person will infect, assuming that the rest of the population is susceptible.

- $R_0 = 1$ is a threshold between epidemic / no epidemic.
- $R_0 > 1$: Disease can invade.
- $R_0 < 1$: Disease will die out.
The basic reproductive ratio $R_0$

- $R_0$ is a key epidemiological measure for how “infectious” a disease is

- E.g. $R_0 = 2$ (Contagion)

In Contagion, Dr. Erin Mears (Kate Winslet) explains $R_0$
Examples for $R_0$

- **BSE**: $0 \leq R_0 \leq 14$
- **Scrapie**: $1.6 \leq R_0 \leq 3.9$
- **Foot & Mouth Disease**: $1.6 \leq R_0 \leq 4.6$
- **Hepatitis C**: $2$
- **Ebola**: $2$
- **HIV**: $4$
- **SARS**: $4$
- **Mumps**: $10$
- **Measles**: $18$

More Contagious
Modelling Epidemics: The SIR model

- $X = \text{nr of susceptibles}$, $Y = \text{nr of infectives}$, $Z = \text{nr of recovered}$
- Describes acute infections transmitted by infected individuals;
- Pathogen causes illness for a period of time followed by death or life-long immunity
We must determine:
• The rate at which susceptible individuals get infected (S→I)
• The rate at which infected individuals recover (or die) (I→R)
This gives rise to 2 model parameters:
• The transmission term $\beta$
• The recovery rate $\gamma$
Transmission rate $S \rightarrow I$

- Depends on the prevalence of infectives, the contact rate and the probability of transmission given contact

**Definition:** Transmission coefficient $\beta = \text{contact rate} \times \text{transmission probability}$

**Definition** Force of infection $\lambda$:
Per capita rate at which susceptible individuals contract the infection

- New infectives are produced at a rate $\lambda \times X$, where $X = nr$ of susceptibles
Frequency versus density dependent transmission

• Force of infection $\lambda$ depends on the number of infectious individuals ($Y(t)$)

• Frequency dependent transmission: $\lambda(t) = \frac{\beta \times Y(t)}{N(t)}$
  • Force of infection depends on frequency of infectives
  • Assumption holds for most human diseases where contact is determined by social constraints rather than population size

• Density dependent transmission: $\lambda(t) = \beta \times Y(t)$
  • Force of infection increases with population size (e.g. individuals crowded in a small space)
  • Assumption appropriate for plant and animal diseases

• The distinction only matters if the population size varies.
  • Otherwise $1/N$ can be absorbed into the parameter $\beta$
The SIR model without demography

- Consider a closed population of constant size $N$
- Model Variables
  - $S(t) = \text{proportion of susceptibles } (X(t)/N)$
  - $I(t) = \text{proportion of infectives } (Y(t)/N)$
  - $R(t) = \text{proportion of recovered } (Z(t)/N)$
- Model Parameters
  - $\beta$: transmission coefficient
  - $\gamma$: recovery rate (the inverse of average infectious period)
The SIR model without demography

• Model equations

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]

With initial conditions

\[
S(t=0) = S(0) \\
I(t=0) = I(0) \\
R(t=0) = R(0)
\]

• Equations describe the rate at which the proportions of susceptible, infectious and recovered individuals change over time

• The model cannot be solved explicitly, i.e. no analytical expression for \( S(t) \), \( I(t) \), \( R(t) \)!
  • Need computer programme

• Constant population size implies \( S(t) + I(t) + R(t) = 1 \) for all times \( t \)
Imagine a scenario where $I_0$ infectives are introduced into a susceptible population.

**Will there be an epidemic?**

- Epidemic will occur if the proportion of infectives increases with time: $\frac{dI}{dt} > 0$
- From the 2$^{nd}$ equation of the SIR model:
  \[
  \frac{dI}{dt} = \beta SI - \gamma I = I (\beta S - \gamma) > 0 \quad \text{only if } S > \gamma/\beta
  \]
- Thus, the infection will only invade if the initial proportion of susceptibles $S_0 > \gamma/\beta$

*What does this result imply for vaccination or other prevention strategies?*
Imagine a scenario where $I_0$ infectives are introduced into a susceptible population. *Will there be an epidemic?*

- The infection will only invade if the initial proportion of susceptibles $S_0 > \frac{\gamma}{\beta}$
- An average infected individual
  - is infectious for a period of $1/\gamma$ days
  - infects $\beta$ susceptible individuals per day
  - will thus generate $\beta \times 1/\gamma$ new infections over its lifetime

$$R_0 = \frac{\beta}{\gamma}$$

- Infection can only invade if $S_0 > \frac{1}{R_0}$
Epidemic burnout

Imagine a scenario where \( I_0 \) infectives are introduced into a susceptible population with \( S_0 > \frac{\gamma}{\beta} \)

What happens in the long-term?

What proportion of the population will contract the infection?

\[
\frac{dS}{dt} = -\beta S I
\]

\[
\frac{dI}{dt} = \beta S I - \gamma I
\]

\[
\frac{dR}{dt} = \gamma I
\]

Solve: \( S(t) = S(0)e^{-R(t)R_0} \)

Given \( R(t) \leq 1 \), this implies that the proportion of susceptibles remains always positive with \( S(t) > e^{-R_0} \) for any time \( t \)

What will stop the epidemic?
Epidemic burnout

What happens in the long-term?
What proportion of the population will contract the infection?

From above: \( S(t) = S(0)e^{-R(t)R_0} \)

At end of epidemic: \( I = 0 \).

Given \( S+I+R=1 \), we can use this equation to calculate the final size \( S(\infty) \) of the epidemic (\( R(\infty) = 1 - S(\infty) \)):

\[
S(\infty) = S(0)e^{(S(\infty)-1)R_0}
\]

- The above equation can only be solved numerically for \( S(\infty) \). It produces the graph on the right.
- This equation is often used to estimate \( R_0 \).

For \( R_0=2 \), what proportion of the population will eventually get infected if everybody is initially susceptible?
Dynamic behaviour

• The exact time profiles depend on the model parameters and on the initial conditions $S(0)$, $I(0)$, $R(0)$

• See Tutorial 1 for investigating the impact of these on prevalence profiles

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]
The SIR model with demography

• Assume the epidemic progresses at a slower time scale so that the assumption of a closed population is no longer valid

• Assume a natural host lifespan of \( \frac{1}{\mu} \) years and that the birth rate is similar to the mortality rate \( \mu \) (i.e. population size is constant)

Generalized SIR model equations:

\[
\frac{dS}{dt} = \mu - \beta S I - \mu S \\
\frac{dI}{dt} = \beta S I - \gamma I - \mu I \\
\frac{dR}{dt} = \gamma I - \mu R
\]

With initial conditions:

\( S(t=0) = S(0) \)

\( I(t=0) = I(0) \)

\( R(t=0) = R(0) \)

\[
\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0
\]
The SIR model with demography

• Transmission rate per infective is $\beta$
• Each infective individual spends on average $\frac{1}{\gamma + \mu}$ time units in class I

\[ R_0 = \frac{\beta}{\gamma + \mu} \]

• $R_0$ is always smaller than for a closed population
**Equilibrium state**

**What is the final outcome of the infection?**

- The disease will eventually settle into an equilibrium state \((S^*, I^*, R^*)\) where

\[
\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0
\]

- Setting the SIR model equations to zero leads to 2 equilibria (outcomes):

\((S^*, I^*, R^*) = (1, 0, 0)\)  \(\text{Disease free equilibirum}\)

\((S^*, I^*, R^*) = \left( \frac{1}{R_0}, \frac{\mu}{\beta} (R_0 - 1), 1 - \frac{1}{R_0} - \frac{\mu}{\beta} (R_0 - 1) \right)\)  \(\text{Endemic equilibrium}\)

**Which outcome will be achieved?**
Equilibrium state

Which outcome will be achieved?
Disease free or endemic equilibrium?

This can be answered with *mathematical stability analysis*:
- Determines for which parameter values a specific equilibrium is stable to small perturbations
- It can be shown that
  - The disease free equilibrium is stable if $R_0 < 1$
  - The endemic equilibrium is stable if $R_0 > 1$

*If an infection can invade (i.e. if $R_0 > 1$), then the topping up of the susceptible pool causes the disease to persist*
Dynamic behaviour

• The SIR model with demography generates damped oscillations:

With some algebra* it can be shown that:

• The period $T \sim \sqrt{\text{mean age of infection} \times \text{mean duration of infectious period}}$

• Mean age of infection $A \approx \frac{L}{R_0 - 1}$ where $L = 1/\mu$ is the average life expectancy

• Measures of $A$ and $L$ lead to estimates for $R_0$

*See e.g. Keeling & Rohani 2008
Adding complexity: The SIRS model

• The SIR model assumes lifelong immunity
• What if this is not the case, i.e. assume immunity is lost at a rate $\alpha$

\[
\frac{dS}{dt} = \mu + \alpha R - \beta SI - \mu S \\
\frac{dI}{dt} = \beta SI - \gamma I - \mu I \\
\frac{dR}{dt} = \gamma I - \alpha R - \mu R
\]

• Loss of immunity does not affect the onset of the disease (same $R_0$!)
• What effect does loss of immunity have on the disease dynamics & equilibrium?

$R_0 = \frac{\beta}{\gamma + \mu}$

See tutorial
Adding complexity: Infections with a carrier state

- Assume a proportion $p$ of infected individuals become chronic carriers.
- These carriers transmit the infection at a reduced rate $\epsilon \beta$, with $\epsilon < 1$.

\[
\begin{align*}
\frac{dS}{dt} &= \mu - (\beta I + \epsilon \beta C)S - \mu S \\
\frac{dI}{dt} &= (\beta I + \epsilon \beta C)S - \gamma I - \mu I \\
\frac{dC}{dt} &= \gamma q I - \mu C \\
\frac{dR}{dt} &= \gamma (1 - q) I - \mu R
\end{align*}
\]

$R_0 = \frac{\beta}{\gamma + \mu} + \frac{p \gamma}{\mu} \frac{\epsilon \beta}{\gamma + \mu}$

- Acutely infecteds
- Chronic Carriers
Adding complexity: Infections with a carrier state

- Assume a proportion $p$ of infected individuals become chronic carriers.
- These carriers transmit the infection at a reduced rate $\varepsilon \beta$, with $\varepsilon < 1$.

$$R_0 = \frac{\beta}{\gamma + \mu} + \frac{p \gamma}{\mu} \frac{\varepsilon \beta}{(\gamma + \mu)}$$

- Acutely infecteds
- Chronic Carriers

- Asymptomatic chronic carriers can cause underestimation of $R_0$.
- What effect does the presence of chronic carriers have on the disease dynamics & equilibrium? 
  ➢ See tutorial.
Summary

• Epidemics can be represented by compartmental ODE models
• Even the simplest epidemiological models require computer algorithms to estimate prevalence profiles
  • But criteria for invasion and for equilibrium conditions can be derived analytically
• The basic reproductive ratio $R_0$ is a key epidemiological measure affecting criteria for invasion extinction and size of the epidemic
• An infection experiences deterministic extinction if $R_0 < 1$
• In the absence of demography, strongly immunizing infections will always go extinct eventually and not all individuals will have become infected
• In the SIR model with demography, the endemic equilibrium is feasible if $R_0 > 1$. Prevalence curves approach this equilibrium through damped oscillations.
References


