Quantitative Genetics, Prediction and Selection Theory

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SELECTION INDICES

BLUP and REML

BAYESIAN INFERENCE

GENOMIC SELECTION

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SELECTION INDICES

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GENOMIC SELECTION

Selection indices

1. Selection indices for a single trait

Indices with information from relatives

Indices with information from several traits

2. Selection indices for multiple traits

Economic Weights

Indices for multiple traits with individual information

Interlude: The Bulmer effect

Indices for multiple traits with family information

3. Some features of selection indices

Selection indices for a single trait

SELECTION BASED ON RELATIVES

I = **LINEAR COMBINATION OF PHENOTYPES FROM RELATIVES**

Infinitesimal model

$$I = \hat{a} = \mathbf{b}'\mathbf{y} = \left[b_1, b_2, \cdots, b_n\right]\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} + e = b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$

The data are centred

Selection indices for a single trait

$$\hat{a} = \mathbf{b}'\mathbf{y}$$

$$a = \mathbf{b}'\mathbf{y} + e$$

$$a \cdot \mathbf{y}' = \mathbf{b}'\mathbf{y}\mathbf{y}' + e \cdot \mathbf{y}'$$

The data are centred

$$E\left(a \cdot \mathbf{y}'\right) = E\left(\mathbf{b}'\mathbf{y}\mathbf{y}'\right) + E\left(e \cdot \mathbf{y}'\right) = \mathbf{b}'E\left(\mathbf{y}\mathbf{y}'\right) + \operatorname{cov}\left(e, \mathbf{y}'\right) = \mathbf{b}'\mathbf{V} + \mathbf{0} = \mathbf{b}'\mathbf{V}$$

$$\mathbf{c}' = \mathbf{b}'\mathbf{V} \quad \rightarrow \quad \mathbf{b}' = \mathbf{c}'\mathbf{V}^{-1} \quad \rightarrow \quad \hat{a} = I = \mathbf{b}'\mathbf{y} = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}$$

$$\boxed{I = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}}$$

$$\mathbf{c}' = E\left(a \cdot \mathbf{y}'\right) = \operatorname{cov}\left(a, \mathbf{y}'\right) = E\left(\left[a \cdot y_1, a \cdot y_2, \cdots, a \cdot y_n\right]\right) =$$
$$\left[\operatorname{cov}\left(a, y_1\right), \operatorname{cov}\left(a, y_2\right), \cdots, \operatorname{cov}\left(a, y_n\right)\right] = \left[\operatorname{cov}\left(a, a_1\right), \operatorname{cov}\left(a, a_2\right), \cdots, \operatorname{cov}\left(a, a_n\right)\right]$$

$$y_i = a_i + e_i$$

$$\mathbf{V} = E\left(\mathbf{y} \cdot \mathbf{y}'\right) = E\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}\left[y_1, y_2, \cdots, y_n\right] = \begin{bmatrix} \sigma_P^2 & \operatorname{cov}\left(y_1, y_2\right) & \cdots & \operatorname{cov}\left(y_1, y_n\right) \\ & \sigma_P^2 & \cdots & \operatorname{cov}\left(y_2, y_n\right) \\ & & \cdots & \cdots \\ & & & \sigma_P^2 \end{bmatrix}$$

Selection indices for a single trait

$$\hat{a} = \mathbf{b}'\mathbf{y}$$

$$a = \mathbf{b}'\mathbf{y} + e$$

$$e = a - \hat{a} = a - \mathbf{b}'\mathbf{y}$$

$$\sigma_e^2 = \mathrm{var}\left(a - \mathbf{b}'\mathbf{y}\right) = \sigma_A^2 + \mathbf{b}'\,\mathrm{var}\left(\mathbf{y}\right)\mathbf{b} - 2\mathbf{b}'\,\mathrm{cov}\left(a, \mathbf{y}\right) = \sigma_A^2 + \mathbf{b}'\mathbf{V}\mathbf{b} - 2\mathbf{c}'\mathbf{b}$$

$$\min\left[Risk\left(\hat{a}\right)\right] \rightarrow \min\left(\sigma_e^2\right)$$

$$\frac{\partial \sigma_e^2}{\partial \mathbf{b}} = 2\mathbf{b}'\mathbf{V} - 2\mathbf{c}' = 0 \quad \rightarrow \quad \hat{\mathbf{b}}' = \mathbf{c}'\mathbf{V}^{-1} \quad \rightarrow \quad I = \hat{a} = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}$$

$$\mathrm{cov}\left(e, \mathbf{y}'\right) = \mathrm{cov}\left(a - \mathbf{b}'\mathbf{y}, \mathbf{y}'\right) = \mathrm{cov}\left(a, \mathbf{y}'\right) - \mathrm{cov}\left(\mathbf{b}'\mathbf{y}, \mathbf{y}'\right) =$$

$$= \mathbf{c}' - \mathbf{b}'\mathbf{V} = \mathbf{c}' - \mathbf{c}'\mathbf{V}^{-1}\mathbf{V} = \mathbf{0}$$

Selection indices for a single trait

$$I = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}$$

$\rho_{A,C}$: **PRECISION**

$\rho^2_{A,C}$: **RELIABILITY**

$$\rho_{A,I} = \frac{\mathrm{cov}(a,I)}{\sigma_A \sigma_I} = \frac{\sigma_I^2}{\sigma_A \sigma_I} = \frac{\sigma_I}{\sigma_A}$$

$$\boxed{\rho^2_{A,I} = \frac{\sigma_I^2}{\sigma_A^2}}$$

$$\mathrm{cov}(a,I) = \mathrm{cov}(a,\mathbf{c}'\mathbf{V}^{-1}\mathbf{y}) = \mathbf{c}'\mathbf{V}^{-1} \cdot \mathrm{cov}(a,\mathbf{y}) = \mathbf{c}'\mathbf{V}^{-1}\mathbf{c} = \sigma_I^2$$

$$\sigma_I^2 = \mathrm{var}(\mathbf{c}'\mathbf{V}^{-1}\mathbf{y}) = \mathbf{c}'\mathbf{V}^{-1} \cdot \mathrm{var}(\mathbf{y}) \cdot \mathbf{V}^{-1}\mathbf{c} = \mathbf{c}'\mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\mathbf{c} = \mathbf{c}'\mathbf{V}^{-1}\mathbf{c}$$

σ_e^2: **Prediction error variance**

$$\sigma_e^2 = \sigma_A^2\left(1 - \rho^2_{A,I}\right) = \sigma_A^2 - \sigma_I^2$$

$CI(95\%)$

$$IC_{95\%} = \pm 2\sigma_e$$

R : **RESPONSE**

$$R = \rho_{A,I} \cdot i \cdot \sigma_A = i \cdot \sigma_I$$

Selection indices for a single trait

EXAMPLE

Index for litter size based on data from
the female (I), two half-sisters (HS) and the mother (M)

$$\mathbf{y}' = \left[y_I, y_{HS}, y_{HS}, y_M\right]$$

$$\sigma_P^2 = 9 \qquad h^2 = 0.10 \qquad \sigma_A^2 = 0.90$$

$$I = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}$$

$$\mathbf{c}' = \operatorname{cov}\left(a_I, \mathbf{y}'\right) = \left[\operatorname{cov}\left(a_I, a_I\right), \operatorname{cov}\left(a_I, a_{HS}\right), \operatorname{cov}\left(a_I, a_{HS}\right), \operatorname{cov}\left(a_I, a_M\right)\right] =$$

$$= \left[\sigma_A^2, \frac{1}{4}\sigma_A^2, \frac{1}{4}\sigma_A^2, \frac{1}{2}\sigma_A^2\right] = \left[0.90, 0.23, 0.23, 0.45\right]$$

$$\mathbf{V} = \begin{bmatrix} \operatorname{var}(y_I) & \operatorname{cov}(y_I, y_{HS}) & \operatorname{cov}(y_I, y_{HS}) & \operatorname{cov}(y_I, y_M) \\ & \operatorname{var}(y_{HS}) & \operatorname{cov}(y_{HS}, y_{HS}) & \operatorname{cov}(y_{HS}, y_M) \\ & & \operatorname{var}(y_{HS}) & \operatorname{cov}(y_{HS}, y_M) \\ & & & \operatorname{var}(y_M) \end{bmatrix} = \begin{bmatrix} 9 & 0.23 & 0.23 & 0.45 \\ & 9 & 0.23 & 0 \\ & & 9 & 0 \\ & & & 9 \end{bmatrix}$$

Selection indices for a single trait

EXAMPLE

$$\hat{a} = I = [0.9, 0.23, 0.23, 0.45]\begin{bmatrix} 9 & 0.23 & 0.23 & 0.45 \\ & 9 & 0.23 & 0 \\ & & 9 & 0 \\ & & & 9 \end{bmatrix}^{-1}\begin{bmatrix} y_I \\ y_{HS} \\ y_{HS} \\ y_M \end{bmatrix} = 0.10\,y_I + 0.02y_{HS} + 0.02\,y_{HS} + 0.05\,y_M$$

$$\sigma_I^2 = \mathbf{c}'\mathbf{V}^{-1}\mathbf{c} = [0.9, 0.23, 0.23, 0.45]\begin{bmatrix} 9 & 0.23 & 0.23 & 0.45 \\ & 9 & 0.23 & 0 \\ & & 9 & 0 \\ & & & 9 \end{bmatrix}^{-1}\begin{bmatrix} 0.90 \\ 0.23 \\ 0.23 \\ 0.45 \end{bmatrix} = 0.12$$

$$\rho_{A,I} = \frac{\sigma_I}{\sigma_A} = \sqrt{\frac{0.12}{0.90}} = 0.36 \;\rightarrow\; \rho_{A,I}^2 = 0.36^2 = 0.13$$

$$\sigma_e^2 = \sigma_A^2 - \sigma_I^2 = 0.90 - 0.12 = 0.78 \;\rightarrow\; IC_{95\%} = \pm 2\sigma_e = \pm 2\sqrt{0.78} = \pm 1.77$$

$$R = i \cdot \sigma_I = 0.8 \cdot \sqrt{0.12} = 0.28 \quad \text{piglets/generation}$$

Selection indices for a single trait

$$\boxed{y_I = 14, \quad y_{HS} = 12, \quad y_{HS} = 13, \quad y_M = 15, \quad m = 10}$$

$$I = \hat{a} = 0.10(14-10) + 0.02(12-10) + 0.02(13-10) + 0.05(15-10) = 0.75$$

$$\hat{a} \pm IC_{95\%} = 0.75 \pm 1.77 = [-1.02 \quad 2.52]$$

$$m + \hat{a} \pm IC_{95\%} = 10.75 \pm 1.77 = [8.98 \quad 12.52]$$

With individual selection

$$R = h^2 \cdot i \cdot \sigma_P = 0.10 \cdot 0.8 \cdot 3 = 0.24$$

$$\rho = h = \sqrt{h^2} = \sqrt{0.10} = 0.32 \rightarrow \rho^2 = 0.10$$

$$\sigma_e^2 = h^2 \sigma_P^2 \left(1 - \rho_{A,C}^2\right) = 0.10 \cdot 3^2 (1 - 0.10) = 0.81$$

$$IC_{95\%} = \pm 2\sqrt{0.81} = \pm 1.80$$

$$\hat{a} \pm IC_{95\%} = 0.10(14-10) \pm 1.8 = 0.4 \pm 1.8 = [-1.4 \quad 2.2]$$

With the selection index

$$R = 0.28$$

$$\rho = 0.36 \rightarrow \rho^2 = 0.13$$

$$\sigma_e^2 = 0.78$$

$$IC_{95\%} = \pm 1.77$$

$$\hat{a} \pm IC_{95\%} = 0.75 \pm 1.77 = [-1.02 \quad 2.52]$$

Selection indices

3.1. Selection indices for a single trait

3.2. Selection indices for multiple traits

3.2.1. Economic weights

3.2.2. The selection index

3.2.3. Indices for multiple traits with family information

3.3. Observations on selection indices

Selection indices for multiple traits

ECONOMIC WEIGHTS

EXAMPLE

$$B = \text{Returns} - \text{Costs} =$$

$$= p \cdot P_p - \left(c \cdot P_c + C_I + \frac{C_F}{n} \right)$$

$$\boxed{w_i = \frac{\Delta B}{\Delta y_i}}$$

p: commercial weight of the rabbit
n: total number of kits per female
c: feed consumption.
C_H: maintenance costs of the female = 64€
C_I: fixed cost of maintaining an individual
P_c: price in € per kg of feed = 0.3€
P_P: price in € per kg of live weight = 2€.

$$w_p = P_p = 2 \ \frac{\text{€}}{\text{kg liveweight}}$$

$$w_c = -P_c = -0.3 \ \frac{\text{€}}{\text{kg feed}}$$

$$w_n = \frac{C_H}{41} - \frac{C_H}{40} = \frac{64}{41} - \frac{64}{40} = -0.039 \ \frac{\text{€}}{\text{kits/doe}}$$

Economic weights should be recalculated frequently

Selection indices for multiple traits

ECONOMIC WEIGHTS

$$B = f\left(y_1, y_2, \ldots y_n\right) \approx \left[\frac{\partial f}{\partial y_1}\right]_{\mathbf{m}} y_1 + \left[\frac{\partial f}{\partial y_2}\right]_{\mathbf{m}} y_2 + \cdots + \left[\frac{\partial f}{\partial y_n}\right]_{\mathbf{m}} y_n$$

$$= \quad w_1 \cdot y_1 + w_2 \cdot y_2 + \cdots + w_n \cdot y_n = \mathbf{w}'\mathbf{y}$$

EXAMPLE

$$B = \text{Returns} - \text{Costs} = p \cdot P_p - \left(c \cdot P_c + C_I + \frac{C_F}{n}\right)$$

p: commercial weight of the rabbit
n: total number of kits per female
c: feed consumption.
C_H: maintenance costs of the female
C_I: fixed cost of maintaining an individual

Selection indices for multiple traits

ECONOMIC WEIGHTS

EXAMPLE

$$B = \text{Returns} - \text{Costs} =$$

$$= p \cdot P_p - \left(c \cdot P_c + C_I + \frac{C_F}{n} \right)$$

p: commercial weight of the rabbit
n: total number of kits per female
c: feed consumption.
C_F: maintenance costs of the female = 64€
C_I: fixed cost of maintaining an individual
P_c: price in € per kg of feed = 0.3€
P_P: price in € per kg of live weight = 2€.

$$w_p = \left[\frac{\partial B}{\partial p} \right]_{\mathbf{m}} = \left[P_p \right]_{\mathbf{m}} = P_p = 2 \ \frac{\text{€}}{\text{kg liveweight}}$$

$$w_c = \left[\frac{\partial B}{\partial c} \right]_{\mathbf{m}} = \left[-P_c \right]_{\mathbf{m}} = -P_c = -0.3 \ \frac{\text{€}}{\text{kg feed}}$$

$$w_n = \left[\frac{\partial B}{\partial n} \right]_{\mathbf{m}} = \left[-\frac{C_F}{n^2} \right]_{\mathbf{m}} = -\frac{C_H}{\bar{n}^2} - \frac{64}{40^2} = -0.04 \ \frac{\text{€}}{\text{kits/doe}}$$

Selection indices for multiple traits

INDEX WITH INDIVIDUAL INFORMATION

SELECTION objective: economic additive value

$$a_E = w_1 a_1 + \cdots + w_m a_m = \mathbf{w}'\boldsymbol{a}$$

I =LINEAR COMBINATION OF PHENOTYPES FROM TRAITS

The data are centred

$$a = \mathbf{b}'\mathbf{y} + e = [b_1, b_2, \cdots, b_n]\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} + e = b_1 y_1 + b_2 y_2 + \cdots + b_n y_n + e$$

SELECTION objectives: $a_1, a_2, \ldots, a_m$
SELECTION criteria: $y_1, y_2, \ldots, y_n$

The OBJECTIVES and CRITERIA may coincide or be wholly or partially different

$$\text{units}(a_E) = \text{units}(w_L a_L + w_P a_P) = \frac{€}{\text{kg milk}} \cdot \text{kg milk} + \frac{€}{\text{kg protein}} \cdot \text{kg protein} = €$$

Selection indices for multiple traits

INDEX WITH INDIVIDUAL INFORMATION

SELECTION objective: economic additive value

OBJECTIVES: a_M, a_P
CRITERIA: y_M, y_P

$$a_E = w_M a_M + w_P a_P$$

$$I = \hat{a}_E = b_M y_M + b_P y_P$$

The data are centred

$$I = \mathbf{c}'\mathbf{V}^{-1}\mathbf{y}$$

$$\hat{a}_M = \mathbf{c}'_M \mathbf{V}^{-1}\mathbf{y}$$

$$\hat{a}_P = \mathbf{c}'_P \mathbf{V}^{-1}\mathbf{y}$$

$$\mathbf{c}'_L = \left[\mathrm{cov}(a_M, a_M), \mathrm{cov}(a_M, a_P)\right] = \left[\sigma^2_{A_M}, \mathrm{cov}(a_M, a_P)\right]$$

$$\mathbf{c}'_P = \left[\mathrm{cov}(a_P, a_M), \mathrm{cov}(a_P, a_P)\right] = \left[\mathrm{cov}(a_M, a_P), \sigma^2_{A_P}\right]$$

$$\hat{\boldsymbol{a}} = \begin{bmatrix} \hat{a}_M \\ \hat{a}_P \end{bmatrix} = \begin{bmatrix} \mathbf{c}'_M \\ \mathbf{c}'_P \end{bmatrix} \mathbf{V}^{-1}\mathbf{y} = \mathbf{C}'\mathbf{V}^{-1}\mathbf{y}$$

$$\mathbf{V} == \begin{bmatrix} \sigma^2_{P_M} & \mathrm{cov}(y_M, y_P) \\ & \sigma^2_{P_P} \end{bmatrix}$$

$$I = \hat{a}_E = w_M \hat{a}_M + w_P \hat{a}_P = \mathbf{w}'\hat{\boldsymbol{a}} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y}$$

Selection indices for multiple traits

EXAMPLE

SELECTION objectives: Feed conversion ratio **FCR** and Backfat thickness **BF**
SELECTION criteria: Growth rate **GR** and Backfat thickness **BF**

ECONOMIC WEIGHTS
0.1 FCR → €3 per animal
1mm BF → 1€ per animal

Heritabilities on the diagonal,
genetic correlations above the diagonal,
phenotypic correlations below the diagonal.

	m	σ_P		FCR	GR	BF
FCR	2.5	0.25	FCR	0.2	–0.7	0.4
GR (g/d)	700	72	GR	–0.6	0.3	0.1
BF (mm)	15	3	BF	0.4	0.2	0.5

$$h^2 = \frac{\sigma_A^2}{\sigma_P^2} \rightarrow \sigma_A^2 = h^2\sigma_P^2$$

$$r_g\left(a_1,a_2\right) = \frac{\mathrm{cov}\left(a_1,a_2\right)}{\sigma_{A_1}\sigma_{A_2}} \rightarrow \mathrm{cov}\left(a_1,a_2\right) = r_g\left(a_1,a_2\right)\cdot\sigma_{A_1}\sigma_{A_2}$$

$$r_P\left(y_1,y_2\right) = \frac{\mathrm{cov}\left(y_1,y_2\right)}{\sigma_{P_1}\sigma_{P_2}} \rightarrow \mathrm{cov}\left(y_1,y_2\right) = r_P\left(y_1,y_2\right)\cdot\sigma_{P_1}\sigma_{P_2}$$

Selection indices for multiple traits

EXAMPLE

SELECTION objectives: Feed conversion ratio **FCR** and Backfat thickness **BF**
SELECTION criteria: Growth rate **GR** and Backfat thickness **BF**

$$\mathrm{cov}(a_1, y_2) = \mathrm{cov}(a_1 \ , \ a_2 + e_2) = \mathrm{cov}(a_1, a_2)$$

$$\mathbf{C}' = \mathrm{cov}(\mathbf{a}, \mathbf{y}') = \mathrm{cov}\left(\begin{bmatrix} a_{FCR} \\ a_{BF} \end{bmatrix}, \begin{bmatrix} y_{GR} & y_{BF} \end{bmatrix}\right) = \begin{bmatrix} \mathrm{cov}(a_{FCR}, a_{GR}) & \mathrm{cov}(a_{FCR}, a_{BF}) \\ \mathrm{cov}(a_{BF}, a_{GR}) & \sigma^2_{A_{BF}} \end{bmatrix} = \begin{bmatrix} -3.1 & 0.1 \\ 8.4 & 4.5 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \sigma^2_{P_{GR}} & \mathrm{cov}(y_{GR}, y_{BF}) \\ \mathrm{cov}(y_{GR}, y_{BF}) & \sigma^2_{P_{BF}} \end{bmatrix} = \begin{bmatrix} 5184 & 43 \\ 43 & 9 \end{bmatrix} \rightarrow \mathbf{V}^{-1} = \begin{bmatrix} 0.00020 & -0.00096 \\ -0.00096 & 0.1157 \end{bmatrix}$$

$$\mathbf{w}' = [w_{FCR}, w_{BF}] = [-30, -1]$$

$$I = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y} = \begin{bmatrix} -30 & -1 \end{bmatrix} \begin{bmatrix} -3.1 & 0.1 \\ 8.4 & 4.5 \end{bmatrix} \begin{bmatrix} 0.00020 & -0.00096 \\ -0.00096 & 0.1157 \end{bmatrix} \begin{bmatrix} y_{GR} \\ y_{BF} \end{bmatrix} = 0.024 y_{GR} - 0.95 y_{BF}$$

GR=770g/d; m_{GR}=700
BF=15mm; m_{BF}=15

$$\boxed{\hat{a}_E = I = 0.024(770-700) - 0.95(12-15) = 4.53 \text{ €}}$$

Selection indices for multiple traits

INDEX WITH INDIVIDUAL INFORMATION

$$I = \hat{a}_E = w_M \hat{a}_M + w_P \hat{a}_P = \mathbf{w}'\hat{\boldsymbol{a}} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y}$$

$\rho_{A,C}$: **PRECISION** | $\rho^2_{A,C}$: **RELIABILITY** | σ^2_e: **PEV** | $CI(95\%)$ | R : **RESPONSE**

$$\rho_{A,I} = \frac{\sigma_I}{\sigma_{A_E}}$$

$$\rho^2_{A,I} = \frac{\sigma^2_I}{\sigma^2_{A_E}}$$

$$\sigma^2_e = \sigma^2_{A_E} - \sigma^2_I$$

$$IC_{95\%} = \pm 2\sigma_e$$

$$R = i \cdot \sigma_I$$

$$\sigma^2_I = \mathrm{var}\left(\mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y}\right) = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1} \cdot \mathrm{var}\left(\mathbf{y}\right) \cdot \mathbf{V}^{-1}\mathbf{C}\mathbf{w} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\mathbf{C}\mathbf{w} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{C}\mathbf{w}$$

$$\sigma^2_{A_E} = \mathrm{var}\left(\mathbf{w}'\mathbf{a}\right) = \mathbf{w}'\left[\mathrm{var}\left(\boldsymbol{a}\right)\right]\mathbf{w} = \mathbf{w}'\begin{bmatrix} \sigma^2_{A_1} & \mathrm{cov}\left(a_1, a_2\right) & \cdots & \mathrm{cov}\left(a_1, a_n\right) \\ & \sigma^2_{A_2} & \cdots & \mathrm{cov}\left(a_2, a_n\right) \\ & & \vdots & \vdots \\ & & & \sigma^2_{A_n} \end{bmatrix}\mathbf{w}$$

Selection indices for multiple traits

INDEX WITH INDIVIDUAL INFORMATION

$$I = \hat{a}_E = \mathrm{w}_M \hat{a}_M + w_P \hat{a}_P = \mathbf{w}'\hat{\boldsymbol{a}} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y}$$

R **: RESPONSE FOR EACH OBJECTIVE**

$$R = i \cdot \sigma_I$$

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1} \begin{bmatrix} \mathrm{cov}(a_1, \mathbf{y}) \\ \mathrm{cov}(a_2, \mathbf{y}) \\ \vdots \\ \mathrm{cov}(a_n, \mathbf{y}) \end{bmatrix} \cdot \frac{i}{\sigma_I} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{C} \cdot \frac{i}{\sigma_I}$$

Selection indices for multiple traits

EXAMPLE

$$\sigma_I^2 = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{C}\mathbf{w} = \begin{bmatrix} -30 & -1 \end{bmatrix} \begin{bmatrix} -3.1 & 0.1 \\ 8.4 & 4.5 \end{bmatrix} \begin{bmatrix} 0.00020 & -0.00096 \\ -0.00096 & 0.1157 \end{bmatrix} \begin{bmatrix} -3.1 & 8.4 \\ 0.1 & 4.5 \end{bmatrix} \begin{bmatrix} -30 \\ -1 \end{bmatrix} = 9.2 \rightarrow \sigma_I = \sqrt{9.2} = 3.03€$$

$$\sigma_{A_E}^2 = \begin{bmatrix} a_{FCR} & a_{GD} \end{bmatrix} \begin{bmatrix} \sigma_{A_{FCR}}^2 & \operatorname{cov}(a_{FCR}, a_{BF}) \\ \operatorname{cov}(a_{FCR}, a_{BF}) & \sigma_{A_{BF}}^2 \end{bmatrix} \begin{bmatrix} a_{FCR} \\ a_{BF} \end{bmatrix} = \begin{bmatrix} -30 & -1 \end{bmatrix} \begin{bmatrix} 0.013 & 0.1 \\ 0.1 & 4.5 \end{bmatrix} \begin{bmatrix} -30 \\ -1 \end{bmatrix} = 22 \rightarrow \sigma_A = \sqrt{22} = 4.69€$$

$$\rho_{A_E,I} = \frac{\sigma_I}{\sigma_{A_E}} = \sqrt{\frac{9.2}{22}} = 0.64 \qquad \rho_{A_E,I}^2 = 0.64^2 = 0.41 \qquad R = i \cdot \sigma_I = 1.3 \cdot 3.03 = 3.94€$$

$$\sigma_e^2 = \sigma_{A_E}^2 - \sigma_I^2 = 22 - 9.2 = 12.8 \quad \rightarrow \quad IC_{95\%} = \pm 2\sigma_e = \pm 2\sqrt{12.8} = \pm 7.2$$

$$\begin{bmatrix} R_{FCR} \\ R_{BF} \end{bmatrix} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{C}\frac{i}{\sigma_I} = \begin{bmatrix} -30 & -1 \end{bmatrix} \begin{bmatrix} -3.1 & 0.1 \\ 8.4 & 4.5 \end{bmatrix} \begin{bmatrix} 0.00020 & -0.00096 \\ -0.00096 & 0.1157 \end{bmatrix} \begin{bmatrix} -3.1 & 8.4 \\ 0.1 & 4.5 \end{bmatrix} \cdot \frac{1.3}{3.03} = \begin{bmatrix} -0.072 \\ -1.7 \end{bmatrix}$$

Interlude:

Changes in genetic parameters with selection. The Bulmer effect

Changes due to selection

$$m = \sum_{1}^{n} m_i = n \cdot k(p - q) = nk(p - 1 + p) = nk(2p - 1)$$

$$m' - m = nk(2p' - 1) - nk(2p - 1) = 2nk(p' - p)$$

$$k \square \frac{1}{\sqrt{n}}$$

$$p' - p \square \frac{1}{\sqrt{n}}$$

$$\sum_{1}^{n} \sigma_A^2 = n \cdot 2pqk^2 = 2np(1 - p)k^2 = 2n\left(p - p^2\right)k^2$$

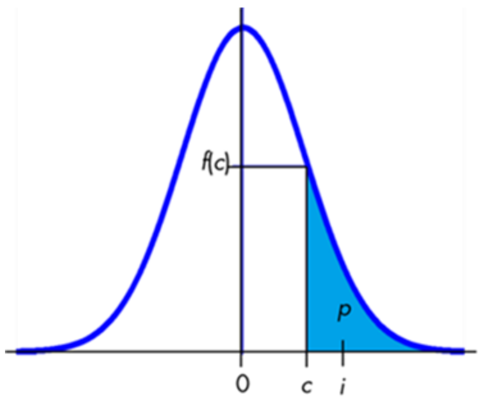
$$\left[\sum_{1}^{n} \sigma_A^2\right]_{\mathrm{G1}} - \left[\sum_{1}^{n} \sigma_A^2\right]_{\mathrm{G0}} = 2n\left(p' - p'^2\right)k^2 - 2n\left(p - p^2\right)k^2 = 2nk^2\left[(p' - p) + p^2 - p'^2\right] \square\ 0$$

$$\frac{d\sigma_A^2}{dt} = 2\sum_{1}^{n} p_i\left(m_i - \overline{m}\right)^3 \square\ 0$$

When $n \to \infty$ distribution $\to$ *Normal*

Assymetry $\to$ 0

Changes due to selection



V^*: variance of selected parents

V : population variance = 1

$$V^* = E(x-i)^2 = E(x^2) - i^2 = \frac{1}{p}\int_c^\infty x^2 f(x)\,dx - i^2 = 1 - i(i-c) < 1 = V$$

$$\boxed{V^* < V}$$

$$\left[\sum_1^n \sigma_A^2\right]_{\mathrm{G1}} - \left[\sum_1^n \sigma_A^2\right]_{\mathrm{G0}} \;\square\; 0$$

Negative covariances

$$\sigma_A^2 = \sigma_{A_1}^2 + \sigma_{A_2}^2 + \sigma_{A_3}^2 + \cdots + 2\operatorname{cov}(A_1, A_2) + 2\operatorname{cov}(A_1, A_3) + \cdots$$

Linkage disequilibrium

	BB	Bb	bb	
AA	50	42	6	98
Aa	20	44	20	84
aa	2	10	6	18
	72	96	32	200

$$frec(\mathrm{A}) = p = \frac{98 + \frac{84}{2}}{200} = 0.70$$

$$frec(\mathrm{B}) = r = \frac{72 + \frac{96}{2}}{200} = 0.60$$

$$frec(\mathrm{AB}) = \frac{50 + \frac{42}{2} + \frac{20}{2} + \frac{44}{4}}{200} = 0.46$$

$$frec_{EQ}(\mathrm{AB}) = p \cdot r = 0.7 \cdot 0.6 = 0.42$$

$$D(\mathrm{AB}) = frec(\mathrm{AB}) - frec_{EQ}(\mathrm{AB}) = 0.46 - (0.7 \cdot 0.6) = 0.04 = D$$

$$D(\mathrm{Ab}) = frec(\mathrm{Ab}) - frec_{\mathrm{EQ}}(\mathrm{Ab}) = 0.24 - (0.7 \cdot 0.4) = -0.04 = -D$$

$$D(\mathrm{aB}) = frec(\mathrm{aB}) - frec_{\mathrm{EQ}}(\mathrm{aB}) = 0.14 - (0.3 \cdot 0.6) = -0.04 = -D$$

$$D(\mathrm{ab}) = frec(\mathrm{ab}) - frec_{\mathrm{EQ}}(\mathrm{ab}) = 0.16 - (0.3 \cdot 0.4) = 0.04 = D$$

Linkage disequilibrium

$$\begin{vmatrix} a & x \\ b & y \end{vmatrix} = \begin{vmatrix} a & a+x \\ b & b+y \end{vmatrix}$$

$$\begin{vmatrix} f_{AB} & f_{Ab} \\ f_{aB} & f_{ab} \end{vmatrix} = \begin{vmatrix} f_{AB} & f_{AB}+f_{Ab} \\ f_{aB} & f_{aB}+f_{ab} \end{vmatrix} = \begin{vmatrix} f_{AB} & p \\ f_{aB} & q \end{vmatrix} = \begin{vmatrix} f_{AB} & p \\ f_{AB}+f_{aB} & p+q \end{vmatrix} = \begin{vmatrix} f_{AB} & p \\ r & 1 \end{vmatrix} = f_{AB} - pr$$

$$\begin{vmatrix} f_{AB} & f_{Ab} \\ f_{aB} & f_{ab} \end{vmatrix} = \begin{vmatrix} f_{AB}+f_{Ab} & f_{Ab} \\ f_{aB}+f_{ab} & f_{ab} \end{vmatrix} = \begin{vmatrix} p & f_{Ab} \\ q & f_{ab} \end{vmatrix} = \begin{vmatrix} p+q & f_{Ab}+f_{ab} \\ q & f_{ab} \end{vmatrix} = \begin{vmatrix} 1 & s \\ q & f_{ab} \end{vmatrix} = f_{ab} - qs$$

$$\boxed{f_{AB} - pr = f_{ab} - qs}$$

$$\boxed{f_{Ab} - ps = f_{aB} - qr}$$

Bulmer effect

$x \backslash y$	$y_1 = k$ $frec(y_1) = r$ **B**	$y_2 = 0$ $frec(y_2) = s$ **b**
$x_1 = k$ $frec(x_1) = p$ **A**	$x_1 \cdot y_1 = k^2$ $frec(x_1, y_1) = f_{AB}$ **AB**	$x_1 \cdot y_2 = 0$ $frec(x_1, y_2) = f_{Ab}$ **Ab**
$x_2 = 0$ $frec(x_2) = q$ **a**	$x_2 \cdot y_1 = 0$ $frec(x_2, y_1) = f_{aB}$ **aB**	$x_2 \cdot y_2 = 0$ $frec(x_2, y_2) = f_{ab}$ **ab**

$$\text{Mean}(x) = \overline{x} = k \cdot p + 0 \cdot q = k \cdot p$$

$$\text{Mean}(y) = \overline{y} = k \cdot r + 0 \cdot s = k \cdot r$$

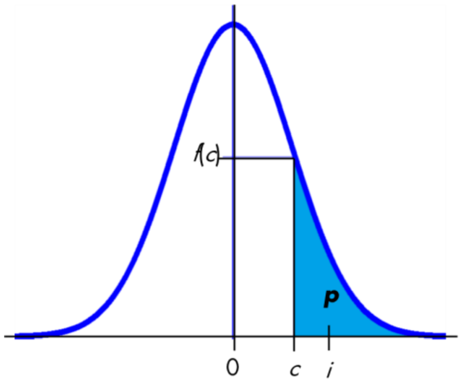
$$\text{cov}(x, y) = \sum x_i y_j f_{ij} - \overline{x} \cdot \overline{y} = k \cdot k \cdot f_{AB} + 0 + 0 + 0 - kp \cdot kr = k^2 (f_{AB} - pr)$$

Bulmer effect

$f_{AB} = 0.6$ $\quad f_{Ab} = 0.2$ $\quad f_{aB} = 0.2$ $\quad f_{ab} = 0$

$$p = f_{AB} + f_{Ab} = 0.6 + 0.2 = 0.8 \qquad q = 1 - p = 0.2$$
$$r = f_{AB} + f_{aB} = 0.6 + 0.2 = 0.8 \qquad s = 1 - r = 0.2$$

$$\text{cov}(x,y) = k^2 \left(f_{AB} - pr\right) = k^2 \left(0.6 - 0.8 \cdot 0.8\right) = -0.04 \cdot k^2$$



$$\boxed{\begin{aligned} D(\text{AB}) &= f_{AB} - p \cdot r = 0.04 \\ D(\text{Ab}) &= f_{Ab} - p \cdot s = -0.04 \\ D(\text{aB}) &= f_{aB} - q \cdot r = -0.04 \\ D(\text{ab}) &= f_{ab} - q \cdot s = \ 0.04 \end{aligned}}$$

$$\boxed{f_{AB} - pr = f_{ab} - qs}$$

$$\rightarrow f_{ab} < q \cdot s \ \rightarrow f_{AB} < p \cdot r \rightarrow \text{cov}(x,y) = k^2 \left(f_{AB} - pr\right) < 0$$

End of the Interlude

Selection indices for multiple traits

INDEX WITH FAMILY INFORMATION

$$I = \hat{a}_E = w_{FCR}\hat{a}_{BF} + w_{FCR}\hat{a}_{BF} = \mathbf{w}'\hat{\mathbf{a}} = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y}$$

$$\boldsymbol{y}' = \left[y_{GR}, y_{BF}, y_{GR}^{HS}, y_{BF}^{HS} \right]$$

$$\mathbf{C}' = \mathrm{cov}\left(\begin{bmatrix} a_{FCR} \\ a_{BF} \end{bmatrix}, \left[y_{GR}, y_{BF}, y_{GR}^{HS}, y_{BF}^{HS} \right] \right) = \begin{bmatrix} \mathrm{cov}(a_{FCR}, a_{GR}) & \mathrm{cov}(a_{GR}, a_{BF}) & \frac{1}{4}\mathrm{cov}(a_{FCR}, a_{GR}) & \frac{1}{4}\mathrm{cov}(a_{FCR}, a_{BF}) \\ \mathrm{cov}(a_{BF}, a_{GR}) & \sigma^2_{A_{BF}} & \frac{1}{4}\mathrm{cov}(a_{BF}, a_{GR}) & \frac{1}{4}\sigma^2_{A_{BF}} \end{bmatrix}$$

$$\mathbf{V} = \mathrm{var}\left(\left[y_{GR}, y_{BF}, y_{GR}^{HS}, y_{BF}^{HS} \right] \right) = \begin{bmatrix} \sigma^2_{P_{GR}} & \mathrm{cov}(y_{GR}, y_{BF}) & \frac{1}{4}\sigma^2_{A_{GR}} & \frac{1}{4}\mathrm{cov}(a_{GR}, a_{BF}) \\ & \sigma^2_{P_{BF}} & \frac{1}{4}\mathrm{cov}(a_{BF}, a_{GR}) & \frac{1}{4}\sigma^2_{A_{BF}} \\ & & \sigma^2_{P_{GR}} & \mathrm{cov}(y_{GR}, y_{BF}) \\ & & & \sigma^2_{P_{BF}} \end{bmatrix}$$

Selection indices for multiple traits

EXAMPLE

$$\mathbf{C}' = \begin{bmatrix} -3.1 & 0.1 & -0.78 & 0.025 \\ 8.4 & 4.5 & 2.1 & 1.1 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 5184 & 43 & 389 & 2.1 \\ & 9 & 2.1 & 1.1 \\ & & 5184 & 43 \\ & & & 9 \end{bmatrix}$$

$$\hat{a}_E = I = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{y} = 0.024 y_{GR} - 0.934\, y_{BF} + 0.004\, y_{GR}^{HS} - 0.114 y_{BF}^{HS}$$

$$\sigma_I^2 = \mathbf{w}'\mathbf{C}'\mathbf{V}^{-1}\mathbf{C}\mathbf{w} = 9.31$$

$$\rho_{A_E,I} = \frac{\sigma_I}{\sigma_{A_E}} = \sqrt{\frac{9.31}{22}} = 0.65$$

$$\rho_{A_E,I}^2 = 0.65^2 = 0.42$$

$$\sigma_e^2 = \sigma_{A_E}^2 - \sigma_I^2 = 22 - 9.31 = 12.7 \quad \rightarrow \quad IC_{95\%} = \pm 2\sigma_e = \pm 2\sqrt{12.7} = \pm 7.1$$

$$R = i \cdot \sigma_I = 1.3 \cdot \sqrt{9.31} = 3.97€$$

Selection indices

- It is possible to improve a trait (target) using other traits as selection criteria.
- It is possible to select indices with restrictions on a trait (which is not to be altered).
- It is possible to select indices with desired gains in certain traits.
- It is possible that some individuals may have missing data, either wholly or partially.
- Some traits may have repeated measurements, differing between individuals.
- Some individuals may have data from more relatives than others.
- Indices do not improve all traits; they enhance overall genetic gain.
- Indices are the optimal selection method if the genetic parameters are the true values. Economic weights are more robust to estimation errors.
- Indices do not usually comprise many objectives or many selection criteria.